

MATHEMATICS teacher

NOVEMBER 2013



beginning algebra

FOCUS
ISSUE

teaching key concepts

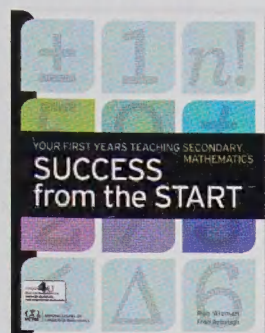
Marygrove College Library
8425 West McNichols Road
Detroit, MI 48221

Fall Savings on All NCTM Publications!

NCTM Members **Save 25%!** Use code **MT1113** when placing order. Offer expires 12/31/2013.*

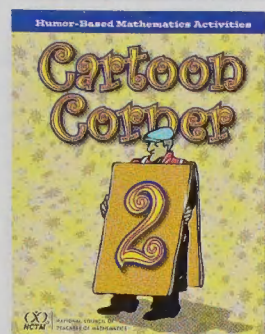
MATH IS ALL AROUND US | MATH IS ALL AROUND US | MATH IS ALL AROUND US | MATH IS ALL AROUND US | MATH IS ALL AROUND US

NEW MATHEMATICS EDUCATION BOOKS!



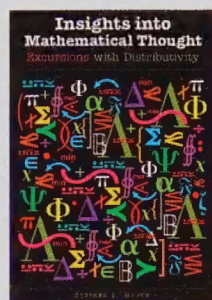
NEW | **Success from the Start: Your First Years Teaching Secondary Mathematics**

BY FRAN ARBAUGH AND ROBERT WIEMAN
©2013, Stock # 13952



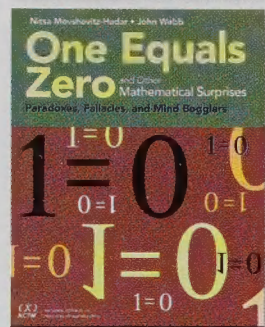
NEW | **Insights into Mathematical Thought: Excursions with Distributivity**

BY STEPHEN I. BROWN
©2013, Stock #14338



NEW | **Second volume of bestselling title!**

Cartoon Corner 2
EDITED BY PEGGY A. HOUSE
©2013, Stock #14373



NEW | **One Equals Zero and Other Mathematical Surprises**

BY NITSA MOVSHOVITZ-HADAR AND JOHN WEBB
Previously published by Key Curriculum Press
©2013, Stock # 14553

MORE NEW BOOKS!

The Science Version of the NCTM's Best-Selling Book

NEW | **5 Practices for Orchestrating Productive Task-Based Discussions in Science**

BY JENNIFER L. CARTER, MARGARET S. SMITH, MARY KAY STEIN, AND DANIELLE K. ROSS
©2013, Stock # 14576

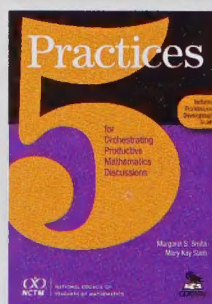
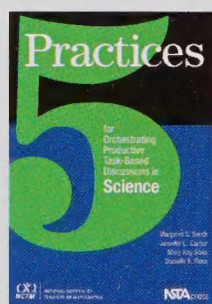
BEST SELLER | **5 Practices for Orchestrating Productive Mathematics Discussions**

BY MARY KAY STEIN AND MARGARET SCHWAN SMITH

[This book] provides teachers with concrete guidance for engaging students in discussions that make the mathematics in classroom lessons transparent to all."

—CATHERINE MARTIN, *Mathematics and Science Director, Denver Public Schools*

©2013, Stock # 13953



NEW BOOKS ON THE COMMON CORE

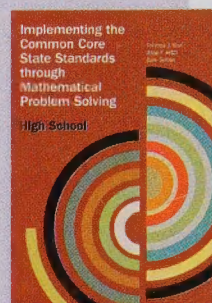
NEW SERIES!

FRANCES CURCIO,
SERIES EDITOR

Implementing the Common Core State Standards through Mathematical Problem Solving: High School

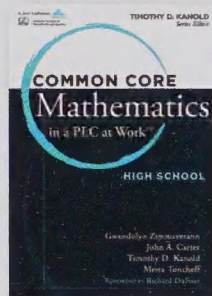
BY THERESA GURL, ALICE ARTZT, AND ALAN SULTAN
©2012, Stock # 14329

Look for more titles in this series to come.



BEST SELLER | **Common Core Mathematics in a PLC at Work, High School**

BY GWEN ZIMMERMAN, JOHN CARTER, TIMOTHY KANOLD, AND MONA TONCHEFF
Copublished with Solution Tree Press
©2013, Stock # 14386



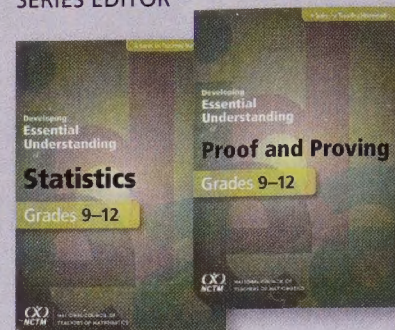
NEW | **Connecting the NCTM Process Standards and the CCSSM Practices**

BY COURTNEY KOESTLER, MATHEW D. FELTON, KRISTEN N. BIEDA, AND SAMUEL OTTEN
©2013, Stock # 14327



New and Best Selling Titles in the Essential Understanding Series

ROSE MARY ZBIEK,
SERIES EDITOR



Developing Essential Understanding: Statistics 9-12

BY ROXY PECK, ROB GOULD, AND STEPHEN MILLER
©2013, Stock #13804

Developing Essential Understanding: Proof and Proving for Teaching Mathematics in Grades 9-12

BY KRISTEN BIEDA, ERIC KNUTH AND AMY ELLIS
©2013, Stock # 13803

Functions for Teaching Mathematics in Grades 9-12

BY THOMAS J. COONEY, GWENDOLYN LLOYD, AND SYBILLA BECKMANN
©2010, Stock #13483

NEW SERIES! | **Putting Essential Understanding into Practice**

BARBARA DOUGHERTY,
SERIES EDITOR

Based on NCTM's Best Selling Developing Essential Understanding Series. You have essential understanding. It's time to put it into practice in your teaching.

Functions in Grades 9-12

BY TERRY CRITES, DAN MEYER, BARBARA DOUGHERTY, AND ROBERT RONAU
© December 2013, Stock #14346

All books available as eBook. *This offer reflects an additional 5% savings off list price, in addition to your regular 20% member discount.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Visit www.nctm.org/catalog for tables of content and sample pages.

For more information or to place an order,
please call (800) 235-7566 or visit www.nctm.org/catalog.



contents

Volume 107, Number 4, November 2013

features

258 **A Skyscraping Feat**

Sarah A. Roberts and Jean S. Lee

Skyscraper Windows, a high cognitive demanding algebra task, addresses the Common Core State Standards for Mathematics.

266 **Cracking Codes and Launching Rockets**

Teo J. Paoletti

This historically significant real-life application of a cryptographic coding technique, which incorporates first-year algebra and geometry, makes mathematics come alive in the classroom.

272 **Connecting Slope, Steepness, and Angles**

Courtney R. Nagle and Deborah Moore-Russo

Students must be able to relate many representations of slope to form an integrated understanding of the concept.

286 **Problems before Procedures: Systems of Equations**

Kasi C. Allen

Students today come to first-year algebra with considerable prior experience and a wide range of skills. Teachers need to modify their instructional strategies accordingly.

292 **Teaching Structure in Algebra**

Ethan M. Merlin

Exercises about glue and trees and addresses can inoculate students against their notorious tendency to reduce incorrectly when simplifying expressions.

298 **Connecting Algebra to Economics**

Victor Mateas

Economics can be an avenue for teaching such algebra concepts as graphing curves, writing linear equations, solving systems of equations, and shifting graphs.

on the cover

The importance of getting students off to a good start in their study of algebra cannot be overemphasized. The *Mathematics Teacher* Editorial Panel chose “Algebra: Teaching the Key Concepts” as the theme for this Focus Issue because of the centrality of algebra in the secondary school mathematics curriculum. The six articles included here, as well as the eleven others to appear in the remaining issues of the volume year, do an admirable job of providing help for teachers who work with students experiencing their first or their n th exposure to algebra. See if you agree.

COVER PHOTO: THINKSTOCK

FEATURE ILLUSTRATIONS: GREGORY ATKINS



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

MATHEMATICS teacher

EDITORIAL PANEL

GREGORY D. FOLEY, Ohio University, Athens, Ohio; *Chair*
DANE R. CAMP, 'Iolani School, Honolulu, Hawaii; *Board of Directors Liaison*
ELIZABETH APPELBAUM, Blue Valley School District, Overland Park, Kansas
DAN CANADA, Eastern Washington University, Cheney; *Digital Liaison*
DAVID CUSTER, Decatur City High School, Decatur, Georgia
STEVE INGRASSIA, Hawken Upper School, Gates Mills, Ohio
ALISON LANGSDORF, Weston High School, Weston, Massachusetts
SANDRA R. MADDEN, University of Massachusetts–Amherst
LAURIE H. RUBEL, Brooklyn College, CUNY, New York
GREG STEPHENS, Hastings High School, Hastings-on-Hudson, New York

JOURNALS STAFF

KEN KREHBIEL, *Associate Executive Director for Communications*
JOANNE HODGES, *Senior Director of Publications*
ALBERT GOETZ, *Journal Editor*
GRETCHEN SMITH MUI, *Copy Editor*
SHEILA J. BARKER, *Review Services Assistant*
CHRISTINE A. NODDIN, *Publications Assistant*
LUANNE M. FLOM, PAMELA A. GRAINGER, ELIZABETH M. SKIPPER,
SARA-LYNN GOPALKRISHNA, *Contributing Editors*

ADVERTISING STAFF

JENNIFER J. JOHNSON, *Senior Director, Member Services, Marketing, and Business Development*
TOM PEARSON, *Sales Manager*
KIM KELEMEN, *National Sales Manager, The Townsend Group*
Kkelemen@townsend-group.com; (301) 215-6710, ex. 103

NCTM BOARD OF DIRECTORS

LINDA M. GOJAK, John Carroll University, University Heights, Ohio; *President*
DIANE J. BRIARS, Pittsburgh, Pennsylvania; *President-Elect*
ROBERT M. DOUCETTE, NCTM; *Executive Director*
ROBERT Q. BERRY III, University of Virginia, Charlottesville
MARGARET (PEG) CAGLE, Vanderbilt University, Nashville, Tennessee
DANE R. CAMP, 'Iolani School, Honolulu, Hawaii
MARK W. ELLIS, California State University, Fullerton
FLORENCE GLANFIELD, University of Alberta, Edmonton
KAREN J. GRAHAM, University of New Hampshire, Durham
GLADIS KERSAINT, University of South Florida, Tampa
LATRENDIA KNIGHTEN, East Baton Rouge Parish School System, Louisiana
RUTH HARBIN MILES, Falmouth Elementary School, Stafford, Virginia
JANE PORATH, Traverse City East Middle School, Michigan
JONATHAN (JON) WRAY, Howard County Public Schools, Maryland
ROSE MARY ZBIEK, Pennsylvania State University, University Park

To contact a journal staff person, e-mail mt@nctm.org.

Mission Statement: The National Council of Teachers of Mathematics is the public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

Mathematics Teacher, an official NCTM journal, is devoted to improving mathematics instruction for grades 8–14 and supporting teacher education programs. It provides a forum for sharing activities and pedagogical strategies, deepening understanding of mathematical ideas, and linking mathematics education research to practice. *Mathematics Teacher* solicits submissions from high school mathematics teachers, university mathematicians, and mathematics educators and strongly encourages manuscripts in which ideas relate to classroom practice. Manuscripts previously published in other journals or books are not acceptable. NCTM publications present a variety of viewpoints. The views expressed or implied in this journal, unless otherwise noted, should not be interpreted as official NCTM positions. The appearance of advertising in NCTM's publications and on its websites in no way implies endorsement of approval by NCTM of any advertising claims or of the advertiser, its product, or its services. NCTM disclaims any liability whatsoever in connection with advertising appearing in NCTM's publications and on its websites.

All correspondence should be addressed to the *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Manuscripts should be prepared according to the *Chicago Manual of Style* and the United States Metric Association's *Guide to the Use of the Metric System*. No author identification should appear on the manuscript; the journal uses a blind-review process. To send submissions, access mt.msubmit.net. Send letters to the editor to mt@nctm.org.

Permission to photocopy material from *Mathematics Teacher* is granted to persons who wish to distribute items individually (not in combination with other articles or works), for educational purposes, in limited quantities, and free of charge or at cost; to librarians who wish to place a limited number of copies on reserve; to authors of scholarly papers; and to any party wishing to make one copy for personal use. Permission must be obtained to use journal material for course packets, commercial works, advertising, or professional development purposes. Uses of journal material beyond those outlined above may violate U.S. copyright law and must be brought to the attention of the National Council of Teachers of Mathematics. For a complete statement of NCTM's copyright policy, see the NCTM website, www.nctm.org.

For information on **article photocopies or back issues**, contact the Customer Care Department in the headquarters office.

A cumulative index appears on the NCTM website at www.nctm.org/mt/mt-indexes.htm. The *Mathematics Teacher* is indexed in *Biography Index*, *Contents Pages in Education*, *Current Index to Journals in Education*, *Education Index*, *Exceptional Child Education Resources*, *Literature Analysis of Microcomputer Publications*, *Mathematical Reviews*, *Media Review Digest*, and *Zentralblatt für Didaktik der Mathematik*.

Information is available from the Headquarters Office or online at www.nctm.org/membership regarding the three **other official journals**, *Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of NCTM's services. Dues for individual membership are \$81 (U.S.), which includes \$34 for each *Mathematics Teacher* subscription. Each additional school journal (*Mathematics Teaching in the Middle School* and *Teaching Children Mathematics*) subscription is \$34. Each additional subscription to the *Journal for Research in Mathematics Education* is \$61. Foreign subscribers add \$18 (U.S.) postage for the first journal and \$4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

The *Mathematics Teacher* (ISSN 0025-5769) (IPM 1213431) is published monthly except June and July, with a combined December/January issue, by the National Council of Teachers of Mathematics at 1906 Association Drive, Reston, VA 20191-1502. Periodicals postage paid at Herndon, Virginia, and at additional mailing addresses. POSTMASTER: Send address changes to *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Telephone: (703) 620-9840; orders: (800) 235-7566; fax: (703) 476-2970; e-mail: nctm@nctm.org. © 2013. The National Council of Teachers of Mathematics, Inc. Printed in the U.S.A.

departments

244 From the Editorial Panel
Beginning Algebra: Teaching Key Concepts

245 Reader Reflections

248 Media Clips
The Psychology of Discounting: Something Doesn't Add Up // Math-hattan
Adam Lavallee // Stephen E. J. Armitage
Edited by Louis Lim and Lionel Garrison

252 Mathematical Lens
Shining the Light on Mathematics
The Editors
Edited by Ron Lancaster and Brigitte Bentele

280 November 2013 Calendar and Solutions
Edited by Margaret Coffey and Art Kalish

306 Connecting Research to Teaching
Linked Representations in Algebra:
Developing Symbolic Meaning
*Douglas A. Lapp, Marie Ermete, Natasha Brackett,
and Karli Powell*
Edited by Margaret Kinzel and Laurie Cavey

313 Delving Deeper
An Archimedean Balance:
Polygons on the Head of a Pin
Charles Worrall
*Edited by Maurice Burke, J. Kevin Colligan,
Maria Fung, and Jeffrey J. Wanko*

313

318 For Your Information
Publications

320 The Back Page: My Favorite Lesson
Perfect Sample and Population Mean Formula
Clarence W. Lienhard
Edited by Jennifer Wexler

calls for manuscripts

**271 2015 Focus Issue: Creating
Classroom Communities**

**285 Understanding and Teaching Statistics
and Probability**

**303 Call for Chapters: *Annual Perspectives in
Mathematics Education (APME) 2015***

305 Current Calls for Manuscripts

more4U

Look online for these additional items:

- Nagle and Moore-Russo (p. 272): Customizable activity sheets
- Mateas (p. 298): Customizable activity sheets
- Connecting Research to Teaching (p. 306): Interactive student investigations



Download one of the free apps for your smartphone. Then scan this tag to access www.nctm.org/mt.

in the next issue

Coming in December 2013/January 2014:

- "Angry Birds Mathematics: Parabolas and Vectors," by John H. Lamb
- "Using KenKen to Build Reasoning Skills," by Harold B. Reiter, John Thornton, and G. Patrick Vennebush
- "Problem Analysis: Challenging All Learners," by Katie Garcia and Alicia Davis



Beginning Algebra: Teaching Key Concepts

What is introductory algebra? Why should students be required to learn the concepts of functions inherent in the course? Does algebra truly prepare students for their future?

The *MT* Editorial Panel strongly believes that, through the study of algebra and mathematics generally, students develop into citizens able to reason logically. They learn to use algebra as a language with which to express the quantitative world. In an introductory course, students will hypothesize about properties and determine whether a counterexample exists to disprove each conjecture; distinguish features of graphs on the basis of the properties of each and the equations inherent in each; and justify their thinking and communicate their reasoning to others. The foundational algebraic skills needed to understand statistics and geometry will open doors to career opportunities.

Are we serving the needs of introductory algebra learners? Are we spurring more students toward a successful experience in their future study of mathematics? Are we helping students understand the mathematics concepts and the reasoning required to make sense of their world? Are we incorporating the Common Core Standards in Mathematics within our algebra curriculum? The 2013 Focus Issue of *Mathematics Teacher* contains articles with rich connections that offer classroom activities and purposeful tasks exploring answers to many of these questions.

Connections between algebra and other disciplines are a major emphasis. A classroom discussion and activity will highlight a historical connection in “Cracking Codes and Launching Rockets,”

while “Connecting Algebra to Economics” uses classroom examples and a larger research task to engage students in algebraic concepts through an understanding of supply and demand.

Another theme is connections between algebraic thinking and making sense of problems. “A Skyscraping Feat” presents a high cognitive demand task that allows students various entry points into the solution and shares several elegant strategies. “Problems before Procedures: Systems of Equations” uses problems to inspire conjectures, which help students make sense of problems before solving them in a more formal way.

A third thread is connections within the learning progressions in algebra. “Connecting Slope, Steepness, and Angles” links various representations of slope to form a complete picture of this crucial concept as students experience it over several years. And “Teaching Structure in Algebra” shares examples that allow students to understand the consistent structure of expressions and thus decrease their reliance on algorithms.

Too often, learning entry-level algebra concepts is an insurmountable struggle for students. They see the course content as a set of discrete ideas requiring memorization of algorithms rather than the study of logic, patterns, and connections. Armed with the language of algebra, students can build quantitative reasoning that opens doors to college and career opportunities. We hope that readers take new ideas from these articles and contribute their own successful strategies for supporting all students as they engage in and learn introductory algebra.

DERIVATIVE OF AREA EQUALS PERIMETER EXTENDED

The relationship established by Zazkis, Sinitsky, and Leikin in their article “Derivative of Area Equals Perimeter—Coincidence or Rule?” (MT May 2013, vol. 106, no. 9, pp. 686–92) also holds for a right circular cylinder with base of radius r and height $2r$. In such a cylinder, the volume is $(\pi r^2)(2r) = 2\pi r^3$, the surface area of the two bases is $2\pi r^2$, and the lateral surface area is $4\pi r^2$. Thus, the surface area is $2\pi r^2 + 4\pi r^2 = 6\pi r^2$. Therefore, the derivative of the volume with respect to the radius equals the surface area.

Since the cylinder can be approximated by right prisms with regular polygon bases, it is not surprising that the relationship also works for such prisms. As was done in the article for the cube, we choose the regular n -gon with height $2r$. Then the volume, V , of the prism is $Bh = (nr^2 \cdot \tan \alpha) \cdot (2r) = 2nr^3 \cdot \tan \alpha$. Since $2B = 2(nr^2 \tan \alpha)$, the lateral area is $2rP_n = 2r(2rn \cdot \tan \alpha) = 4r^2n \cdot \tan \alpha$, so $SA = 6nr^2 \cdot \tan \alpha$.

So, again, the derivative of the volume equals the surface area. The relationship between the volume of a sphere and its surface area can also be generalized to n dimensions; see www.sjsu.edu/faculty/watkins/ndim.htm.

An interesting project might include

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to mt@nctm.org. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

extending the relationship to n -dimensional cylinders, prisms, and Platonic solids.

Robert E. Clay

rclay@daltonstate.edu

Dalton State College

Dalton, GA, May 16, 2013

GRAPHING POLAR CURVES

I enjoyed Jonathan Lawes’s fine article “Graphing Polar Curves (MT May 2013, vol. 106, no. 9, pp. 660–67). The process it describes of relating a Cartesian graph and the polar graph is wonderfully enlightening for students.

Inspired by Lawes’s article, I have written a Sine of the Times blog post (<http://blog.keycurriculum.com/2013/04/cartesian-and-polar-graphs>) about using technology to explore this connection, including some questions and challenges that I hope readers may find enjoyable.

Scott Steketee

stek@kcptech.com

KCP Technologies

Philadelphia, May 2, 2013

PROBLEM 19, FEBRUARY 2013 CALENDAR

Problem 19 of the March 2013 Calendar (MT Feb. 2013, vol. 106, no. 6, p. 441) states:

The numerical value of the volume of a cylinder is twice the numerical value of its surface area. If the radius and the height are both integers, determine the smallest possible volume.

MT’s answer stated that one of the integral solutions, $r = 5$ and $h = 20$, gives a volume of 500π .

However, another solution, $r = 6$ and $h = 12$, yields a volume of 432π . The corresponding surface area is 216π . The

calculation of the volume would be $\pi r^2 h = \pi(6^2)(12) = 432\pi$.

Anita Schuloff

aschuloff@paramus-catholic.org

Paramus Catholic High School

Paramus, NJ, Feb. 2, 2013

PROBLEM 24, MARCH 2013 CALENDAR

Problem 24 of the March 2013 Calendar (MT Feb. 2013, vol. 106, no. 6, p. 441) was given as follows:

Circle O has radius 6. Congruent circles P and Q are internally tangent to circle O and externally tangent to each other. Circle S is internally tangent to circle O and externally tangent to circles P and Q . Find the radius of circle S .

The answer given was 2. The first line of the solution stated, “Since the radius of circle O is 6, the radius of circle P is 3.” With that assumption, the answer is indeed 2. But nothing in the problem requires this assumption (or the equivalent assumption that the centers of circles O , P , and Q are collinear).

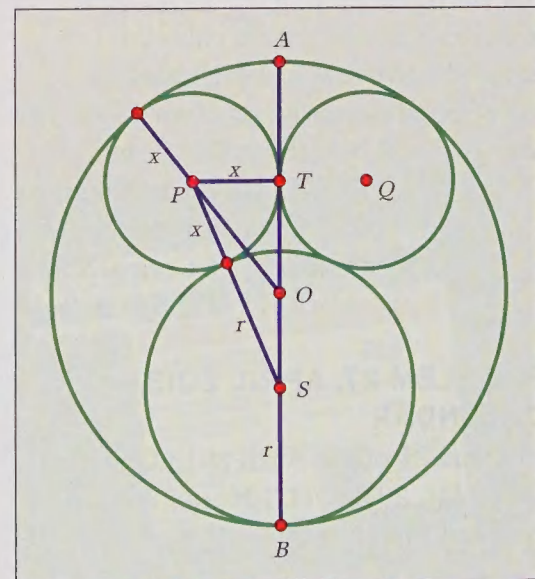


Fig. 1 (Clemens)

Without this assumption, the radius of circle S can be any number $2 \leq \text{radius} < 6$, depending on the radii of the congruent circles P and Q .

In **figure 1** (Clemens, p. 245), $ATOSB$ is a diameter of circle O passing through the center of circle S and the point of tangency, T , of circles P and Q . Since the radius of circle O is 6, we know that $AT + TS + SB = 12$. Letting r be the radius of circle S and x be the radius of circle P , we have the following:

$$\begin{aligned} OP &= 6 - x \\ OT &= \sqrt{(6 - x)^2 - x^2} = \sqrt{36 - 12x} \\ AT &= 6 - OT = 6 - \sqrt{36 - 12x} \\ SP &= r + x \\ TS &= \sqrt{(r + x)^2 - x^2} = \sqrt{r^2 + 2xr} \end{aligned}$$

Substituting the expressions for AT and TS into our original equation and noting that $SB = r$, we get

$$6 - \sqrt{36 - 12x} + \sqrt{r^2 + 2xr} + r = 12.$$

The radicals can be eliminated in the usual way and the resulting equation divided by x (since $x = 0$ does not satisfy the conditions of the problem) to arrive at

$$xr^2 + 24r^2 + 12xr - 144r + 36x = 0.$$

We can then solve for r :

$$r = \frac{72 - 6x + 24\sqrt{9 - 3x}}{x + 24},$$

where the plus sign in the numerator has been chosen to make $r \geq 2$. When x assumes its maximum value of 3, the equation gives $r = 2$, which was the stated solution to the problem. Also, as $x \rightarrow 0$, $r \rightarrow 6$, which makes sense.

Glenn P. Clemens

gclemens@nncsk12.org
Norwood-Norfolk Central School
Norwood, NY, Apr. 5, 2013

PROBLEM 27, APRIL 2013 CALENDAR

Problem 27 of the April 2013 Calendar (MT Mar. 2013, vol. 106, no. 7, p. 521) is stated as follows:

How many unique nets can be used to

form a cube? (A *net* is a pattern or two-dimensional shape that can be folded into a three-dimensional shape.)

The answer is not 11 but, rather, infinitely many.

Start with a cube and pick any point on the top face. Use a scissors to cut from that point to each corner of the top face and then from each corner of the top face down to the base. Unfold the result and flatten the cube.

Because of obvious symmetries, we can pick sets of two or four points on the top that produce the same unfolded shape. However, it is also obvious that there are infinitely many distinct such shapes. The definition of *net* in the problem does not rule out cuts that do not follow edges of the cube.

J. Kevin Colligan

jkcolligan@verizon.net
SRA International (retired)
Columbia, MD, Mar. 18, 2013

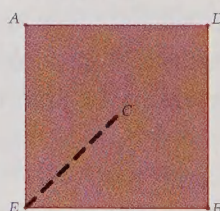
From the Calendar editors: Colligan's interpretation of the problem is interesting. Although there can be more than one definition for the term *polyhedral net*, the standard is that a net of a polyhedron (a planar net) is determined by the edges of the polyhedron.

MAY 2013 CALENDAR PROBLEMS: ALTERNATE SOLUTIONS

Problem 1, May 2013 Calendar

Problem 1 of the May 2013 Calendar (MT Apr. 2013, vol. 106, no. 8, p. 600) stated:

A chocolate cake baked in a 9 in. $\times$ 9 in. square pan is to be divided equally among three people. The three straight cuts will each have one endpoint at the cake's center. If the first cut is made as shown, where should the next two cuts be made?



Let E be the fourth corner of the square $ADBE$ and let Q_1 and Q_2 be the midpoints of AE and BE , respectively. By symmetry, $AP_2 = BP_1 = x$ in. Then the area of the trapezoid AP_2CQ_1 equals the

area of $BP_1CQ_2 = (x + 4.5)(4.5)/2$, and the area of $AP_2CP_1BE = 2 \cdot \text{area } AP_2CQ_1 + \text{area } EQ_1CQ_2 \rightarrow 54 = (x + 4.5)(4.5) + (4.5)(4.5) \rightarrow x = 3$ in.

Problem 10, May 2013 Calendar

Problem 10 of the May 2013 Calendar (MT Apr. 2013, p. 600) was given as follows:

We have

$$\begin{aligned} \sqrt{a} + \sqrt{a} &= \sqrt{b}, \\ \sqrt{b} + \sqrt{b} &= \sqrt{c}, \\ \sqrt{c} + \sqrt{c} &= \sqrt{d}, \dots, \\ \sqrt{g} + \sqrt{g} &= \sqrt{h}. \end{aligned}$$

Find the sum

$$a + b + c + \dots + h$$

in terms of a .

The last formula given in the solution to problem 10 should read $S = a(1 - 4^8)/(1 - 4) = 21845a$.

Problem 11, May 2013 Calendar

Problem 11 of the May 2013 Calendar (MT Apr. 2013, p. 600) stated:

A fraction with a numerator of 1 and a positive integral denominator is called a *unit fraction*. When a fraction is written as a sum of two or more unit fractions, it may be referred to as an *Egyptian fraction*. Write the fraction $13/40$ as a sum of two distinct unit fractions.

The positive factors of 40 are 1, 2, 4, 5, 8, 10, 20, and 40. The only combination of three numerators whose sum is 13 is $1 + 2 + 10$; hence,

$$\frac{13}{40} = \frac{1}{20} + \frac{2}{20} + \frac{10}{20} = \frac{1}{20} + \frac{1}{10} + \frac{1}{2}.$$

Problem 26, May 2013 Calendar

Problem 26 of the May 2013 Calendar (MT Apr. 2013, p. 601) was given as follows:

Points $A(-5, 0)$ and $B(5, 0)$ are the endpoints of a diameter of a circle. Point P moves along the semicircular path from B to A . Find an equation that gives the area of $\triangle APB$ in terms

of the x -coordinate of point P .

Here is a slight modification to the alternate solution: The triangles ACP and BCP are similar, and

$$\frac{PC}{BC} = \frac{AC}{PC} \rightarrow \frac{y}{5+x} = \frac{5-x}{y}.$$

The rest is as in the given solution.

Jeganathan Sriskandarajah
Jsriskandara@madisoncollege.edu
 Madison College
 Madison, WI, Apr. 9, 2013

MORE ON A RARE JULY

Regarding “A Rare July?” (Media Clips, *MT* Mar. 2013, vol. 106, no. 7, pp. 492, 494), table 2 contains some historical inaccuracy. The dates in the table accompanying the article, which include years from 1490 to 1501, are calculated from the Gregorian calendar. However, the Gregorian calendar was not adopted until 1582, well past the dates in the table. Among other discrepancies, 1500 would have been a leap year, and July 1,

Year	Day for July 1
1994	Friday
1995	Saturday
1996	Monday
1997	Tuesday
1998	Wednesday
1999	Thursday
2000	Saturday
2001	Sunday
2002	Monday
2003	Tuesday
2004	Thursday
2005	Friday

Table 1 (Kluepfel)

1490, was not a Friday in the Julian calendar in effect at that time.

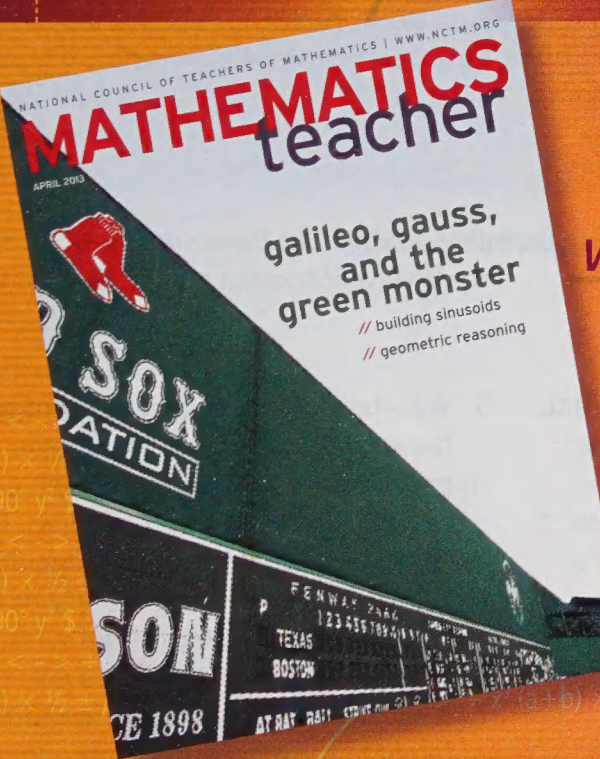
Perhaps July 1, 1490, would have been a Thursday had the Gregorian calendar been extrapolated back to that year. This usage is called the Gregorian

proleptic calendar. As the Gregorian calendar repeats in a cycle of 400 years, repeating the same days of the week for given dates, the dates in a proleptic Gregorian calendar for 1490 would match those in 1890. But July 1, 1890, was a Tuesday, and thus so would be July 1, 1490, in the Gregorian proleptic calendar. Besides, as mentioned, the lack of one of the leap years actually prevents the eleven-year recurrence.

Using a computer program, we can see that the eleven-year time between occurrences of five full weekends in July is more common than if they required a century year. Here is a list of several successive such occasions: 1622–33, 1650–61, 1678–89, 1718–29, 1746–57, . . . , 1966–77, 1994–2005, 2022–33, 2050–61, and 2078–89. To take the most recent example from the article, table 2 should have looked like **table 1 (Kluepfel)**.

Charles Kluepfel
Chasklu@aol.com
 Verizon (retired)
 Bloomfield, NJ, May 13, 2013

INSPIRING TEACHERS. ENGAGING STUDENTS. BUILDING THE FUTURE.



MATHEMATICS
teacher

galileo, gauss,
and the
green monster

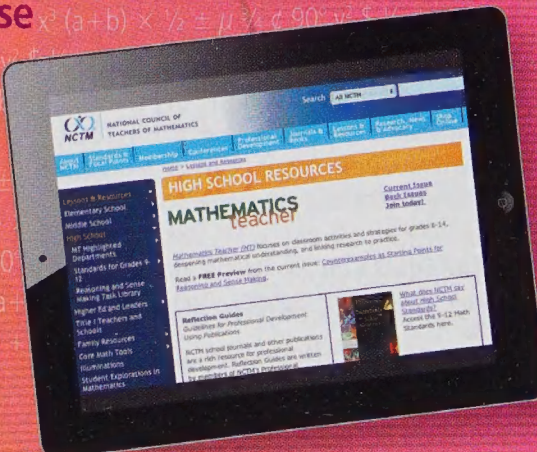
// building sinusoids
// geometric reasoning

Let NCTM Help You Make Your Job Easier!

We have the resources to meet your challenges.

Check out www.nctm.org/high for:

- Lessons and activities
- Problems database
- Core math tools
- Online articles
- Topic resources





**NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS**

(800) 235-7566 | WWW.NCTM.ORG

The Psychology of Discounting: Something Doesn't Add Up

When retailers want to entice customers to buy a particular product, they typically offer it at a discount. According to a new study to be published in the *Journal of Marketing*, they are missing a trick.

A team of researchers, led by Akshay Rao of the University of Minnesota's Carlson School of Management, looked at consumers' attitudes to discounting. Shoppers, they found, much prefer getting something extra free to getting something cheaper. The main reason is that most people are useless at fractions.

Consumers often struggle to realise, for example, that a 50% increase in quantity is the same as a 33% discount in price. They overwhelmingly assume the former is better value. In an experiment, the researchers sold 73% more hand lotion when it was offered in a bonus pack than when it carried an

equivalent discount (even after all other effects, such as a desire to stockpile, were controlled for).

This numerical blind spot remains even when the deal clearly favours the discounted product. In another experiment, this time on his undergraduates, Mr. Rao offered two deals on loose coffee beans: 33% extra free or 33% off the price. The discount is by far the better proposition, but the supposedly clever students viewed them as equivalent.

Studies have shown other ways in which retailers can exploit consumers' innumeracy. One is to befuddle them with double discounting. People are more likely to see a bargain in a product that has been reduced by 20%, and then by an additional 25%, than one that has been subject to an equivalent, one-off, 40% reduction.

Source: "The Psychology of Discounting: Something Doesn't Add Up," *The Economist*, June 30, 2012, <http://www.economist.com/node/21557801>

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit <http://www.nctm.org/publications/content.aspx?id=10440#media>.

Edited by **Louis Lim**, louis.lim1@gmail.com
Thornhill Secondary School, Thornhill, ON, Canada

Lionel Garrison, garrison@horacemann.org
Horace Mann School, Bronx, NY

1. If a 2 oz. candy bar sells for \$3, what is the price per ounce?
2. If the size of the candy bar is increased by 50% and the price remains \$3, what is the price per ounce of the candy bar?
3. If the 2 oz. candy bar is sold at a 50% discount in price, what is the new price?
4. What is the price per ounce of the 2 oz. candy bar at the 50% discounted price?
5. Which discount—50% more candy bar or 50% less price—would you prefer?
6. What percentage price drop would be the same as increasing the size of the candy bar by 50%?
7. The article states, "People are more likely to see a bargain in a product that has been reduced by 20%, and then by an additional 25%. . . ." If an item's price has been reduced by 10% and then by an additional 30%, what is the total percentage discount?

Math-hattan



THINKSTOCK

[Mathematician Glen Whitney, a former hedge fund manager, has devoted himself to the new National Museum of Mathematics, 11 East 26th Street, New York, N.Y. In 2009 he led a math tour of Manhattan that was covered by the *New Yorker Magazine*. Excerpts follow.]

... [Whitney] jaywalked across Sixty-sixth Street to Lincoln Center Plaza, where he pointed to “the best spot in the city for perfect one-point perspective”—a view east through the cloistered walkway on the north side of Avery Fisher Hall. ...

As the tour moved down Broadway, Whitney calculated the advantage of taking the hypotenuse while travelling through a grid (you gain two blocks for every nine). He stopped at a fire hydrant, produced a wrench, and explained the rationale behind pentagonal lug nuts (they are wrench-proof). ...

In Whitney’s view, mathematical ignorance is insidious, and it manifests itself in many ways. “The purest example is the lottery,” he said. “The lottery is a tax on the mathematically illiterate.”

... Whitney stopped off at the National Debt Clock, on Forty-fourth

Street, which tracks the deficit in real time. At that second, the clock read \$11,518,960,404,062, and a second later the number was tens of thousands of dollars higher. The family share of it held steady, at \$96,700. “People just don’t seem to care,” he said. “Why? Because it’s impossible to understand it.” He has tried the analogical approach favored by astronomers. “If Jupiter is the size of a basketball, then Earth is a pea. O.K., so let’s suppose the national debt is a basketball. Your annual salary would be unseen by the most powerful microscope in the world.”

Source: Nick Paumgarten, “Math-Hattan,” *The New Yorker*, August 3, 2009, http://www.newyorker.com/talk/2009/08/03/090803ta_talk_paumgarten#ixzz1SvzLrUBw

1. (a) Verify the statement that “you gain two blocks for every nine” by taking the hypotenuse through a grid. Assume that the blocks are square.
- (b) Estimate the time that you would save walking “across town” in New York City if you could walk in a straight line to your destination instead of following the sidewalks. State all the assumptions that you make in your calculations.
2. Explain why pentagonal lug nuts are wrench-proof.
3. Interpret the statement “the lottery is a tax on the mathematically illiterate.”
4. (a) According to the National Debt Clock, how many families are there in the United States? How accurate is this number?
- (b) How accurate is Whitney’s analogy? Use any resources, including the Internet, to support your answer.
- (c) Suppose that the national debt is a basketball. How big a sphere would represent the annual salary of the average North American? State any assumptions that you make in your calculations.
- (d) Now suppose that the average North American’s annual salary is a pea. How big would the national debt be? State any additional assumptions that you make in your calculations.

"Discounting" answers

1. The price per ounce is \$1.50:

$$\frac{\$3}{2 \text{ oz.}} = \frac{\$1.50}{\text{oz.}}$$

2. The new size will be $1.5 \cdot 2 \text{ oz.} = 3 \text{ oz.}$
The price per ounce is then $\$3/3 \text{ oz.} = \$1/\text{oz.}$

3. To determine the new price, subtract 50% of \$3 from \$3 to obtain \$1.50.

4. For the 2 oz. candy bar that is discounted 50%, the price per ounce is $\$1.50/2 \text{ oz.} = \$0.75/\text{oz.}$

5. The cost per ounce is less when the price is discounted 50%.

6. The new price per ounce would need to be \$1. For a 2 oz. candy bar, the new price must be \$2. If x represents the percentage decrease, then $2 = 3 \cdot (1 - x) \rightarrow x = 1/3$. So a 33% discount in price would be the same as a 50% increase in quantity.

7. If we let p represent the original price of the object, then $0.9p$ represents the price after a 10% discount, and $(0.7) \cdot (0.9p) = 0.63p$ represents the price after the 30% discount. So the effective discount is $p - 0.63p = 0.37 = 37\%$.

In general, if two successive discounts of d_1 and d_2 are applied to an item whose selling price is p , the new price will be $(1 - d_1) \cdot (1 - d_2) \cdot p$. Expanding, we get $(1 - d_1 - d_2 + d_1d_2) \cdot p$. Therefore, the total discount is $1 - (1 - d_1 - d_2 + d_1d_2) = d_1 + d_2 - d_1d_2$. In this problem, $d_1 = 10\%$ and $d_2 = 30\%$, so $d_1 + d_2 - d_1d_2 = 10\% + 30\% - (10\%)(30\%) = 37\%$.

"Math-hattan" answers

1. (a) Let L_1 be the path length from A through C to B, and let L_2 be the

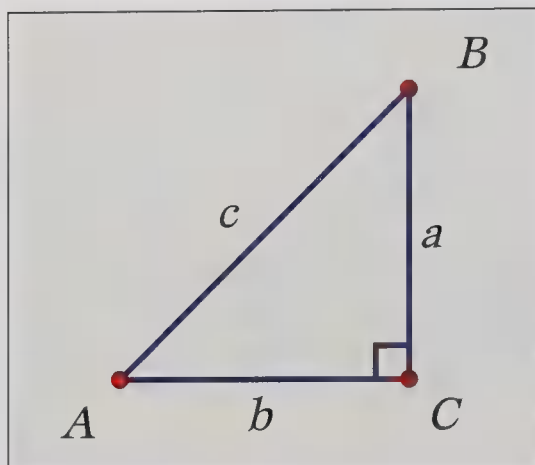


Fig. 1 ("Math-hattan")

path length from A directly to B. If we assume that we are taking the hypotenuse along square blocks, then $a = b$, $L_1 = 2a$, and

$$L_2 = \sqrt{a^2 + b^2} = \sqrt{2}a.$$

Then $L_2/L_1 = \sqrt{2}/2 \rightarrow L_2 = L_1/\sqrt{2}$. So if L_1 is 9 blocks, then $L_2 = 9/\sqrt{2} \approx 6.4$ blocks. Thus, by using the diagonal, we gain about 2.6 blocks each time we walk 9 blocks (see **fig. 1** ["Math-hattan"]). (Note that the Manhattan blocks referred to in the clip are not square but rectangular. Readers interested in pursuing this idea are encouraged to import images of the area from Google Maps™ or similar mapping software into GeoGebra or The Geometer's Sketchpad® and check further.)

- (b) This question has many possible answers. We made the following assumptions:

- The average person can walk at a speed of 6 km/hr.
- "Across town" means across Manhattan.
- The borough of Manhattan is approximately square with side lengths of 7 km, and we are traversing the borough from one corner to the opposite corner.

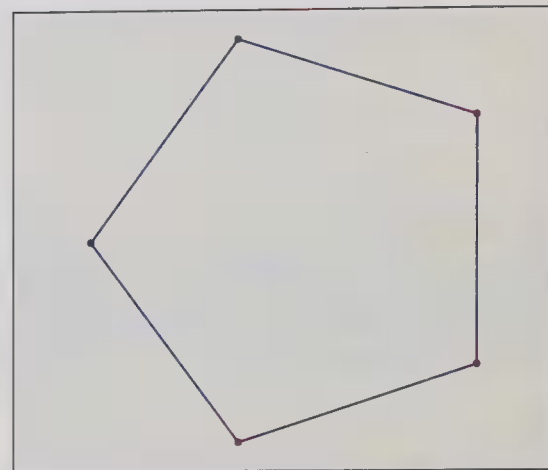


Fig. 2 ("Math-hattan")

With these assumptions, we calculate the time that it would take to walk from one corner of Manhattan to the other along the sidewalks:

$$\Delta t_1 = \frac{L_1}{v} = \frac{14 \text{ km}}{6 \text{ km/hr.}} = \frac{7}{3} \text{ hr.}$$

Now we calculate the time that it would take if we could walk in a straight line all the way to our destination:

$$\Delta t_2 = \frac{L_2}{v} = \frac{(14/\sqrt{2}) \text{ km}}{6 \text{ km/hr.}} \approx 1.65 \text{ hr.}$$

Since $7/3 \text{ hr.} - 1.65 \text{ hr.} \approx 0.68 \text{ hr.}$ and $0.68 \text{ hr.} \approx 41 \text{ min.}$, we would save 41 min. by walking the diagonal across Manhattan instead of along the sidewalks.

2. A wrench requires two parallel surfaces on which to clamp down so that it can apply enough leverage to function properly. As shown in **figure 2** ("Math-hattan"), a pentagon has no parallel sides. Any regular polygon with an odd number of sides would also qualify as wrench-proof.
3. The meaning of the statement "the lottery is a tax on the mathematically illiterate" is that playing the lottery is almost always going to result in

the player losing money. One of the simplest types of lotteries is the 50/50 draw. In this type of lottery, a player pays x dollars for a ticket; if this ticket is drawn, the player wins half of the total proceeds collected from ticket sales. Suppose that you are playing a 50/50 draw, each ticket costs \$2, and 200 tickets are sold. If you buy 1 ticket, then the probability of winning is $1/200 = .005$. The total proceeds collected from the ticket sales are $\$2 \cdot 200 = \400 , so the winning prize is half that, or \$200. Playing the same lottery over time, you would be expected to win, on average, $P(\text{winning}) \cdot \text{payoff} = .005 \cdot \$200 = \$1$. Since you are spending \$2 on each ticket but winning only an average of \$1, you are essentially paying \$1 every time you choose to play the 50/50 draw. If you understand mathematics, then you should expect to lose money when playing the lottery.

4. (a) The number of families can be found by dividing the debt by the family share, so according to the National Debt Clock, there are about

$$\frac{\$11,518,960,404,062}{\$96,700 / \text{family}} \approx 119,120,583 \text{ families}$$

in the United States. Census bureau data from the period up to 2011 indicate that there are around 114 million families in the United States. This error is around 4%.

- (b) Whitney (or the author) has confused the analogy. The equatorial diameters of Jupiter and Earth are, respectively, 141,700 and 12,800 km. The ratio of the diameter of Jupiter to that of Earth is thus $141,700/12,800 \approx 11$; the ratio of their volumes is therefore approximately 11^3 , or around 1324. One way to correct the analogy is to say that if Jupiter is the size of a basketball, then Earth

is the size of a Ping-Pong® ball. What Whitney probably intended to say was that if the sun is the size of a basketball, then Jupiter is the size of a pea.

- (c) We can answer this question using a proportion, but do we use volume units or linear units? We will first compare volumes, but our assumptions will be the same for each case: The radius of a basketball is about 4.75 in., and the average North American salary is \$40,000 per year. If we let r be the radius of an object representing the average annual salary of a North American, then

$$\begin{aligned} \frac{\frac{4}{3}\pi r^3 \text{ in.}^3}{\$40,000} &= \frac{449 \text{ in.}^3}{\$11,518,960,404,062} \rightarrow \\ \frac{4}{3}\pi r^3 &= \frac{17,960,000 \text{ in.}^3}{11,518,960,404,062} \\ &\approx 1.6 \times 10^{-6} \text{ in.} \rightarrow \\ r &\approx 0.007 \text{ in.} \end{aligned}$$

If we use this comparison, the radius of a sphere representing the average North American salary would be about 0.007 in., or about 0.018 cm.

If we compare linear measurements, then

$$\begin{aligned} \frac{r}{\$40,000} &= \frac{4.75 \text{ in.}}{\$11,518,960,404,062} \rightarrow \\ r &= \frac{190,000 \text{ in.}}{11,518,960,404,062} \\ &\approx 1.6 \times 10^{-8} \text{ in.} \end{aligned}$$

Electron microscopes with the best resolution let us see objects that are 4.3 microns in length. One micron is approximately 4×10^{-5} in., so 4.3 microns would be around 1.72×10^{-4} in.

- (c) If we assume that the diameter of a pea is 0.5 cm and if we let D be the diameter of an object representing the national debt, then

$$\begin{aligned} D &= \frac{\$11,518,960,404,062}{\$40,000} \cdot 0.5 \text{ cm} \\ &\approx 143,987,005.1 \text{ cm.} \end{aligned}$$

The diameter of the national debt would be about 900 miles, or almost the distance from Washington, D.C., to Kansas City, Missouri, as the crow flies.



ADAM LAVALLEE, alavallee@episcopalacademy.org, who submitted "Discounting," teaches high school mathematics at Episcopal Academy in Newtown Square, Pennsylvania. He is interested in experiential learning techniques, problem-based learning, and authentic technology use in the classroom.



STEPHEN E. J. ARMITAGE, stephen_armitage@kprdsb.ca, who submitted "Math-hattan," teaches mathematics and physics for the Kawartha Pine Ridge District School Board in Peterborough, Ontario, Canada, and mathematics for the Virtual High School. He is interested in probability, statistics, and their application to game theory.

IAS/Park City Mathematics Institute (PCMI)
June 29 – July 19, 2014
Park City, Utah

Summer School Teachers Program
A three-week summer program for:
High School Teachers
Middle School Teachers
Elementary Coaches/Teachers

Education Theme:
Making Mathematical Connections

Organizers of Summer School Teachers Program:
Gail Burrill, Michigan State University;
James King, University of Washington;
Carol Hattan, Skyview High School, Vancouver, WA.

Do mathematics...
Reflect on practice...
Become a resource...

Applications and information online: pcmi.ias.edu
Deadline: January 31, 2014

IAS/Park City Mathematics Institute
Institute for Advanced Study
Princeton, NJ 08540

Financial Support Available

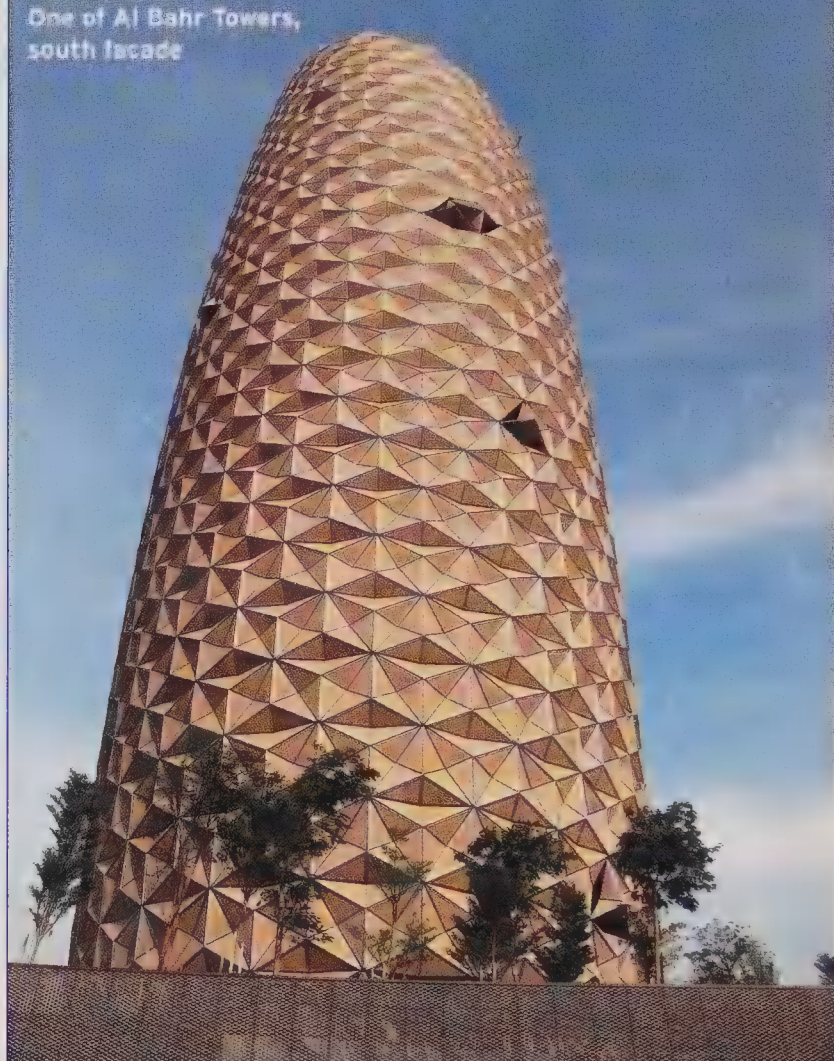
Shining the Light on Mathematics

Photograph 1
One of Al Bahr Towers,
north facade



RON LANCASTER

Photograph 2
One of Al Bahr Towers,
south facade



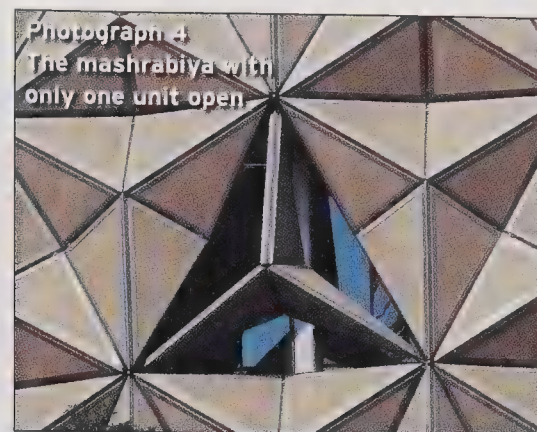
RON LANCASTER

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit <http://www.nctm.org/publications/content.aspx?id=10440#lens>.

Edited by **Ron Lancaster**, ron2718@nas.net, University of Toronto, ON, Canada, and **Brigitte Bentele**, brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY

To set new standards of environmental responsibility, Aedas, the designers of the Al Bahr Towers, the new headquarters of the Abu Dhabi Investment Council (ADIC) (see **photographs 1** and **2**) created “a diaphanous screen that envelopes the most exposed aspect of the building in the form of a dynamic ‘mashrabiya,’ opening and closing in response to the sun’s path, significantly reducing the solar heat gain and providing a more comfortable internal environment,” according to Peter Oborn, Aedas joint managing director

(Leech 2011). The computer-controlled, robotic facade is made up of “more than two-thousand translucent, parasol-like units that will open and close as the sun moves over their surface. In short, the Al Bahr Towers will be clothed in a moving, protective veil.” The mashrabiya, “a 21st century reinterpretation of the carved and perforated screens that traditionally provided shade and privacy to Islamic houses throughout the Middle East,” covers the east, south, and west sides of the two towers.



RON LANCASTER

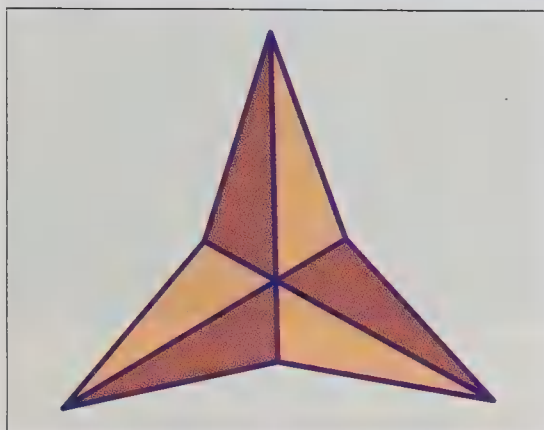


Fig. 1 The ADIC logo can be represented in two dimensions.

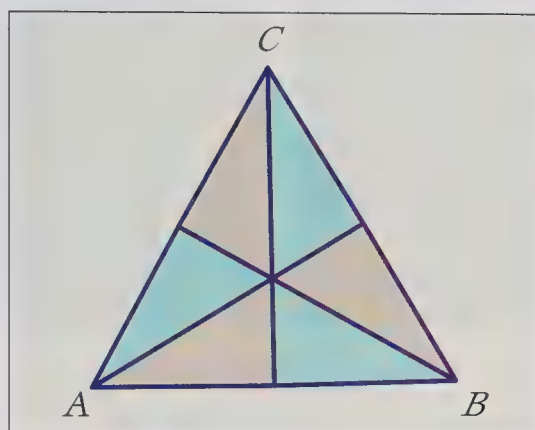


Fig. 2 The equilateral triangle can be separated into six congruent triangles when the center is connected to the vertices and the midpoints of the sides.

1. The ADIC logo is a three-dimensional structure similar to the units on the facade (see **photograph 3**). A two-dimensional representation of the logo, as it probably appears in ADIC's publications, was created using The Geometer's Sketchpad® (see **fig. 1**).

(a) In an equilateral triangle, the *centroid* (the intersection of the triangle's medians), *orthocenter* (intersection of the triangle's altitudes), *circumcenter* (intersection of the perpendicular bisectors of the triangle's sides), and *incenter* (intersection of the triangle's angle bisectors) are all coincident—that is, they occur at the same point. Construct an equilateral triangle using a compass and straightedge. Then find the center by constructing the centroid, orthocenter, circumcenter, or incenter.

(b) If an equilateral triangle is positioned on the coordinate plane so that two of its vertices are $A(-3, 0)$ and $B(3, 0)$, what would be the positive y -coordinate of C , its third vertex?

(c) What are the coordinates of the center of triangle ABC ?

(d) Six congruent triangles are formed when the center of triangle ABC is connected to each vertex and to the midpoint of the sides (see **fig. 2**). If we use the same dimensions as in (c), what are the lengths of the sides of these triangles?

(e) One way of thinking about the three-dimensional logo is to imagine lifting the triangle by its center while keeping the vertices in place, thereby collapsing the sides inward, as if all interior lines in the triangle were hinged (see **photograph 4**). If the ratio of the area of the two-dimensional

logo (see **fig. 1**) to the area of the equilateral triangle is 2:3, by how much has each side been pulled in?

2. The more than 2000 units of the facade open or close to provide shade as the sun rises and sets. Each unit “is constructed from 15 components that form a triangular, Teflon-coated, fiberglass mesh set in an aluminum and stainless steel frame” (Leech 2011). **Photograph 4** shows a close-up of a single open unit among several closed units.

(a) Suppose that, when closed, the unit is a solid triangular pyramid with an equilateral triangle of side 6 m as its base and height 1 m. What is the volume of this solid? What is its lateral surface area?

(b) Suppose that, when closed, each unit of the mashrabiya consisted of the two-dimensional equilateral triangle (see **fig. 2**). As the unit opens, allowing more light to penetrate to the interior of the building, three joined triangular pyramids are formed. Five edges of each pyramid correspond to segments of the equilateral triangle; the sixth edge varies as the unit opens (one of the three pyramids in several stages is shown in **fig. 3**). Each “pyramid” has no

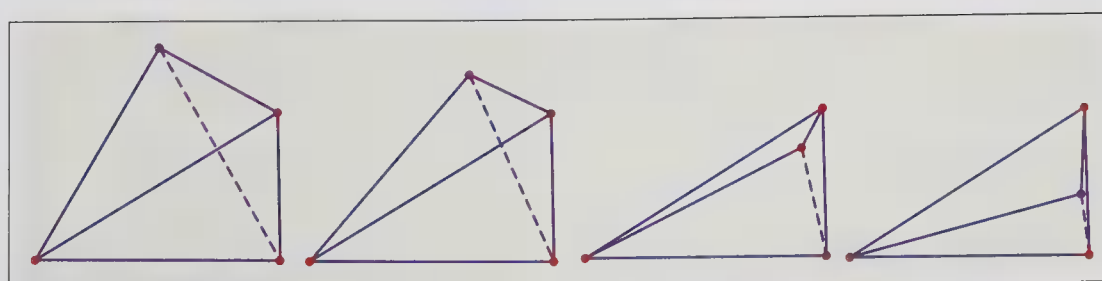
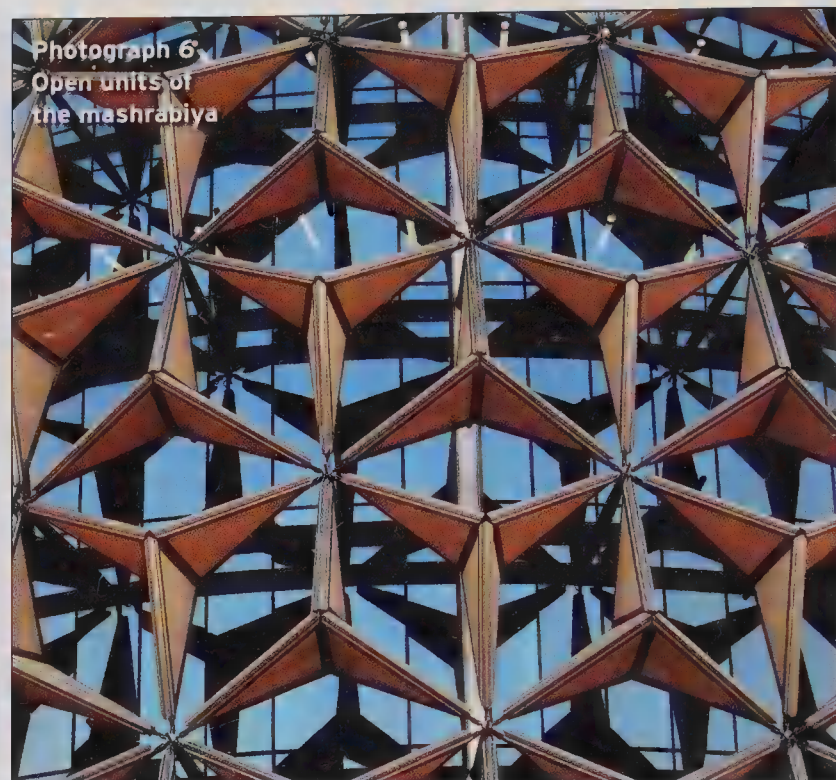


Fig. 3 The dashed line represents the only dimension that changes as the unit opens or closes.



volume when the unit is entirely closed and would have no volume if it were entirely open. When the unit is open so that each pyramid is pulled up to half its maximum amount, what is the length of the varying side of the pyramid?

(c) What is the dihedral angle of the half-opened unit?

(d) Behind each unit is a glass window. When the unit is at the position found in part (b), what percentage of glass is exposed?

3. **Photograph 5** shows the units of the mashrabiya after the sun has risen, and **photograph 6** shows a view of the building after the sun has set. Given its environmental goals, the architectural firm was no doubt interested in the

times of the sunrise and sunset in Abu Dhabi throughout the year.

Figure 4 shows the graphs of the times of the sunrise and sunset in Abu Dhabi from January 1 through December 31, 2013. Note that when we read upward along the numbers, they decrease, creating a small problem when modeling these graphs with equations. For the following questions, use **figure 4b**, which shows the graphs reflected so that the numbers along the y -axis are increasing.

(a) Let x represent the day of the year (with $x = 0$ corresponding to January 1), and let y represent the time of the sunrise. The earliest and the latest sunrise occurred at 5:33 a.m. and 7:07 a.m., respectively. On January 1, the time of the sunrise

was 7:07 a.m. Find an equation of the form $y = a \sin(bx + c) + d$ to model these data.

(b) Let x represent the day of the year (with $x = 0$ corresponding to January 1), and let y represent the time of the sunset. The earliest and latest times of the sunset are 17:46 p.m. and 19:15 p.m., respectively. On January 1, the time of the sunrise was 17:46 p.m. Find an equation of the form $y = a \sin(bx + c) + d$ to model these data.

(c) Use your equations from parts (a) and (b) to find an equation for the amount of daylight as a function of the day of the year.

(d) What days of the year will have more than 12 hours of daylight?

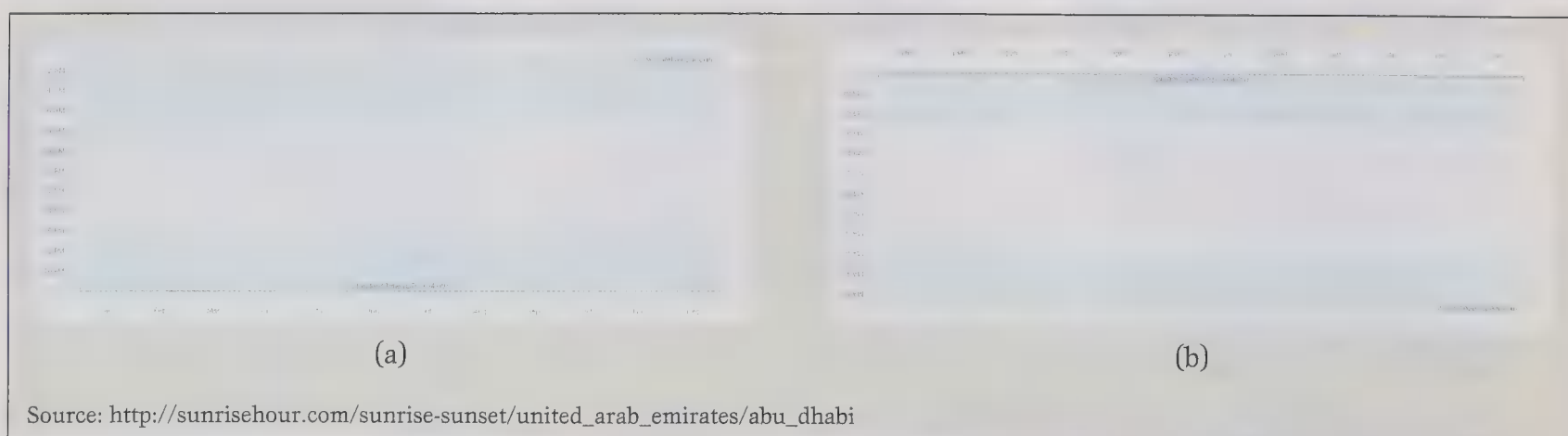


Fig. 4 To make curve fitting easier, reflect the graph shown in (a) so that the units on the vertical axis increase, as shown in (b).

MATHEMATICAL LENS solutions

1. (a) To construct an equilateral triangle, draw a segment with a straightedge. With the compass set at the length of the segment, draw two arcs from the endpoints of the segment. Either intersection of the arcs is the third vertex of the equilateral triangle. To find the point of concurrency, use arcs on two sides or angles to create a perpendicular bisector or an angle bisector, respectively.

$$V = \frac{1}{3} \cdot \frac{1}{2} \cdot 6 \cdot 3\sqrt{3} = 3\sqrt{3} \text{ m}^3$$

To find the surface area, find the lateral height or the height of each triangle. The lateral height is the hypotenuse of a right triangle formed by the height of the pyramid and the perpendicular segment from the center of the equilateral triangle to one of its sides. Therefore,

$$h_l = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

Then the lateral area is $3 \cdot \frac{1}{2} \cdot 6 \cdot 2 = 18 \text{ m}^2$.

- (b) In an equilateral triangle, the side opposite the 60° angle is half the side of the triangle times the square root of 3. Therefore, the y -coordinate of C is $3\sqrt{3}$.

- (c) The center is $(0, \sqrt{3})$.

- (d) The sides of the triangles are $\sqrt{3}$, 3, and $2\sqrt{3}$.

- (e) Let x represent the amount that each side has been pulled in. The area of the equilateral triangle is $s^2\sqrt{3}/4 = 9\sqrt{3}$. From that area, three triangles with base and altitude 6 and x , respectively, have been subtracted. We solve the equation:

$$\begin{aligned} \frac{9\sqrt{3} - 3 \cdot \frac{1}{2} \cdot 6 \cdot x}{9\sqrt{3}} &= \frac{2}{3} \\ \frac{9\sqrt{3} - 9x}{9\sqrt{3}} &= \frac{2}{3} \\ 1 - \frac{x}{\sqrt{3}} &= \frac{2}{3} \\ \frac{x}{\sqrt{3}} &= \frac{1}{3} \rightarrow x = \frac{\sqrt{3}}{3} \end{aligned}$$

2. (a) To find the volume, use the formula $V = Bh/3$, where B represents the area of the base of the pyramid and h is its height:

$$\begin{aligned} BE &= \sqrt{BC^2 - CE^2} = \sqrt{3 - \frac{3}{4}} = \frac{3}{2} \\ AE &= \sqrt{AC^2 - CE^2} = \sqrt{12 - \frac{3}{4}} = \frac{3}{2}\sqrt{5} \end{aligned}$$

Then, using the law of cosines, we find the measure of $\angle BAE$:

$$\begin{aligned} \angle BAE &= \cos^{-1} \left(\frac{3^2 + \left(\frac{3}{2}\sqrt{5}\right)^2 - \left(\frac{3}{2}\right)^2}{2 \cdot 3 \cdot \frac{3}{2} \cdot \sqrt{5}} \right) \\ &\approx 26.565^\circ \end{aligned}$$

We then use the angle measure to find BF :

$$\sin(26.565^\circ) = \frac{BF}{3}; BF \approx 1.342$$

Therefore, BD is approximately 2.684 m.

- (c) The dihedral angle is

$$\angle CAE = \sin^{-1} \left(\frac{\frac{\sqrt{3}}{2}}{\frac{2}{2\sqrt{3}}} \right) = \sin^{-1} \left(\frac{1}{4} \right) \approx 14.478^\circ.$$

- (d) The area of the kite $ABED$ (see **fig. 5**) obstructs the light coming into the building. The maximum amount of light—that is, the area of the equilateral triangle with side 6 meters—is $6^2 \cdot \sqrt{3}/4$. The area of kite $ABED$ is

$$\frac{1}{2} \cdot AE \cdot BD \approx \frac{1}{2} \cdot \frac{3}{2} \sqrt{5} \cdot 2.684 \approx 4.5 \text{ m}^2.$$

With three of these pyramids obstructing the light, the percentage of light entering is

$$\frac{4.5 \cdot 3}{9\sqrt{3}} \approx 86.6\%.$$

- (b) Refer to **figure 5**. From the equilateral triangle with side 6, we have the lengths of sides $AB = 3 = AD$; $BC = \sqrt{3} = CD$; and $AC = 2\sqrt{3}$. When the point C is pulled up to its maximum, BD would equal 0. Point C would be at half its maximum when C is $\sqrt{3}/2$ above the plane ABD . Let E be the projection of C onto plane ABD , and let F be the intersection of AE and BD . Since $\angle BEC$ and $\angle CEA$ are right angles, we can use the Pythagorean theorem to find the lengths of BE and AE :

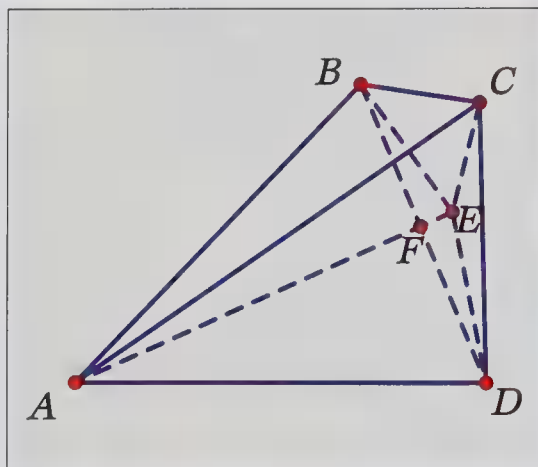


Fig. 5 BD has length of around 2.7 m when the pyramid is at the halfway point.

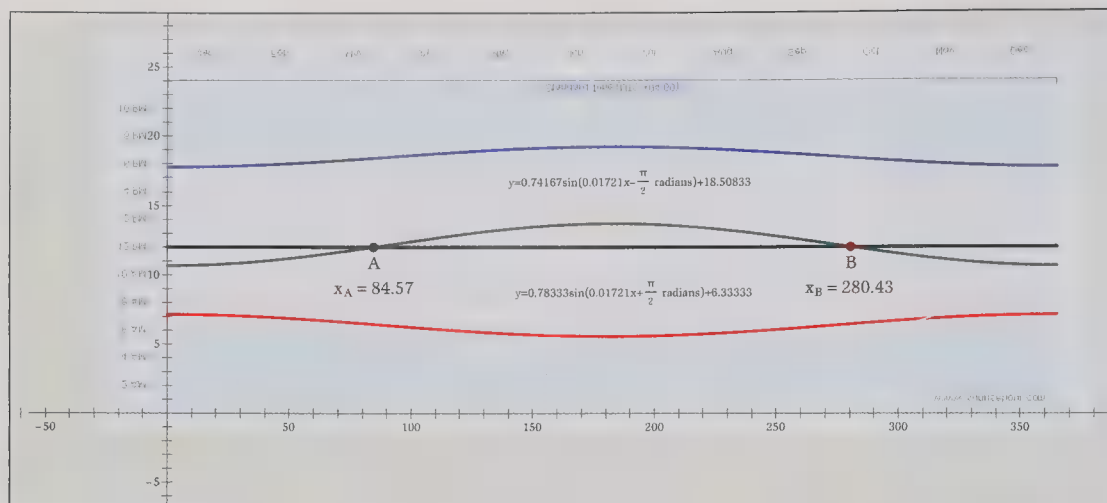


Fig. 6 The intersection points indicate that each day between March 25 and October 7 will have more than 12 hours of daylight.

3. (a)

$$y = 0.78333 \sin\left(0.01721x + \frac{\pi}{2}\right) + 6.33333.$$

To obtain the amplitude of the function, we compute half the difference between the maximum and the minimum, so $a \approx 0.78333$. Since the frequency = $2\pi/\text{period}$, $b = 2\pi/365 \approx 0.01721$. The vertical shift, or sinusoidal axis, is the

average of the maximum and the minimum, so $d \approx 6.33333$. The sine curve is shifted right by $\pi/2 = c$.

(b)

$$y = 0.74617 \sin\left(0.01721x - \frac{\pi}{2}\right) + 18.50833.$$

Analysis similar to that performed in (a) yields the values of a , b , and d ; c remains $\pi/2$. Thus, a is half

the difference of $19 + 15/60$ and $17 + 46/60$; b is $2\pi/365$; and d is the average of $19 + 15/60$ and $17 + 46/60$.

(c) The difference between the two equations found in parts (a) and (b) will provide the amount of daylight as a function of the day of the year:

$$\begin{aligned} y &= 0.74617 \sin\left(0.01721x - \frac{\pi}{2}\right) + 18.50833 \\ &\quad - \left[0.78333 \sin\left(0.01721x + \frac{\pi}{2}\right) + 6.33333\right] \\ &= 0.74617 \sin\left(0.01721x - \frac{\pi}{2}\right) \\ &\quad - 0.78333 \sin\left(0.01721x + \frac{\pi}{2}\right) + 12.175 \end{aligned}$$

(d) The days that have more than 12 hours of daylight can be found by setting the equation from part (c) equal to 12 and solving for x . Alternatively, the intersection points between the graphs of $y = 12$ and the graph of the equation from part (c) can be found using a graphing calculator or software such as The Geometer's Sketchpad (see **fig. 6**). Either method will show that each day between the 84th and the 280th days of the year will have more than 12 hours of daylight.

REFERENCE

Leech, Nick. 2011. "Desert-Smart Towers Carry Their Own Sunscreen." *The National*. Sept. 17. <http://www.thenational.ae/news/uae-news/desert-smart-towers-carry-their-own-sunscreen>.

I  oblate spheroids.

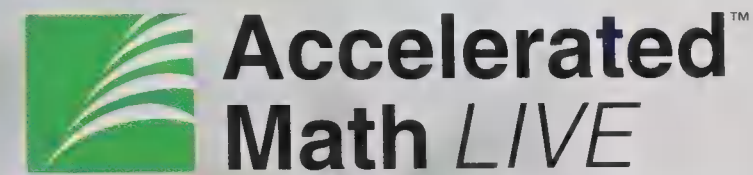
MATHEMATICS
IS ALL AROUND US.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Simplify your path to CCSS success.

Now available...



Bringing Personalized Practice to Life!

Best use of your technology—
flexible & adaptable

Built for the Common Core—
new content for all K12 grades



See what else is NEW!

www.renlearn.com/lp/22391

(800) 338-4204, ref. #22391

RENAISSANCE
LEARNING
Accelerating learning for all

A Skyscraper Feat

Skyscraper Windows, a high cognitive demanding algebra task, addresses the Common Core State Standards for Mathematics.

Sarah A. Roberts and Jean S. Lee

Research shows that the greatest gains in student learning in mathematics classrooms occur in classrooms in which there is sustained use of high cognitive demanding tasks throughout instruction (Boston and Smith 2009). High cognitive demanding tasks, which we will refer to as rich tasks, are mathematics problems that are complex, less structured, and can take longer than other typical problems to solve because they require students to “think about, develop, use and make sense of mathematics” (Stein, Grover, and Henningsen 1996, p. 459). Such tasks also afford students and teachers the opportunity to “make sense of problems and persevere in solving them,” one of the Common Core mathematics standards (CCSSI 2010).

Simply providing students with challenging tasks is not enough; teachers must also be prepared to facilitate student learning by first solving the tasks themselves. Smith and Stein (2011) explain: “Anticipating students’ responses involves developing considered expectations about how students

might mathematically interpret a problem, the array of strategies—both correct and incorrect—that they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn” (p. 8). This article builds on Smith and Stein’s idea of anticipating student solutions for a cognitively demanding algebra task by providing several possible student pathways and teacher responses that allow teachers and students to engage with the Common Core mathematics practice of making sense of problems and persevering in solving them.

We present a cognitively demanding algebra task, Skyscraper Windows, which allows students to consider multiple algebraic representations when solving problems. Our preservice teachers solved this problem and found multiple solution paths that they thought algebra students might take. Using their solutions, we provide suggestions for preparing to implement the task and illustrate possible pathways that students and teachers might take.

ing



IMPLEMENTING THE SKYSCRAPER WINDOWS TASK

The task we present is an adaptation of the Skyscraper Windows task (Driscoll 1999, p. 70):

A building is 12 stories high and is covered entirely with windows on all four sides. Each floor has 38 windows. Once a year, all the windows are washed. The cost for washing the windows is \$2.00 for each first-floor window, \$2.50 for each second-floor window, \$3.00 for each third-floor window, and so on. How much will it cost to wash all the windows of this building? What if the building were 30 stories high? n stories high?

We will illustrate the various levels of mathematical sophistication inherent in this task and identify various goals that a teacher might have when implementing it.

Teachers can take the following steps to implement any rich task, including the Skyscraper Windows problem:

1. Determine which standards the problem aligns with to meet learning objectives.
2. Do the math—solve the problem yourself! Consider multiple solution paths and how the solutions connect to students' prior knowledge.

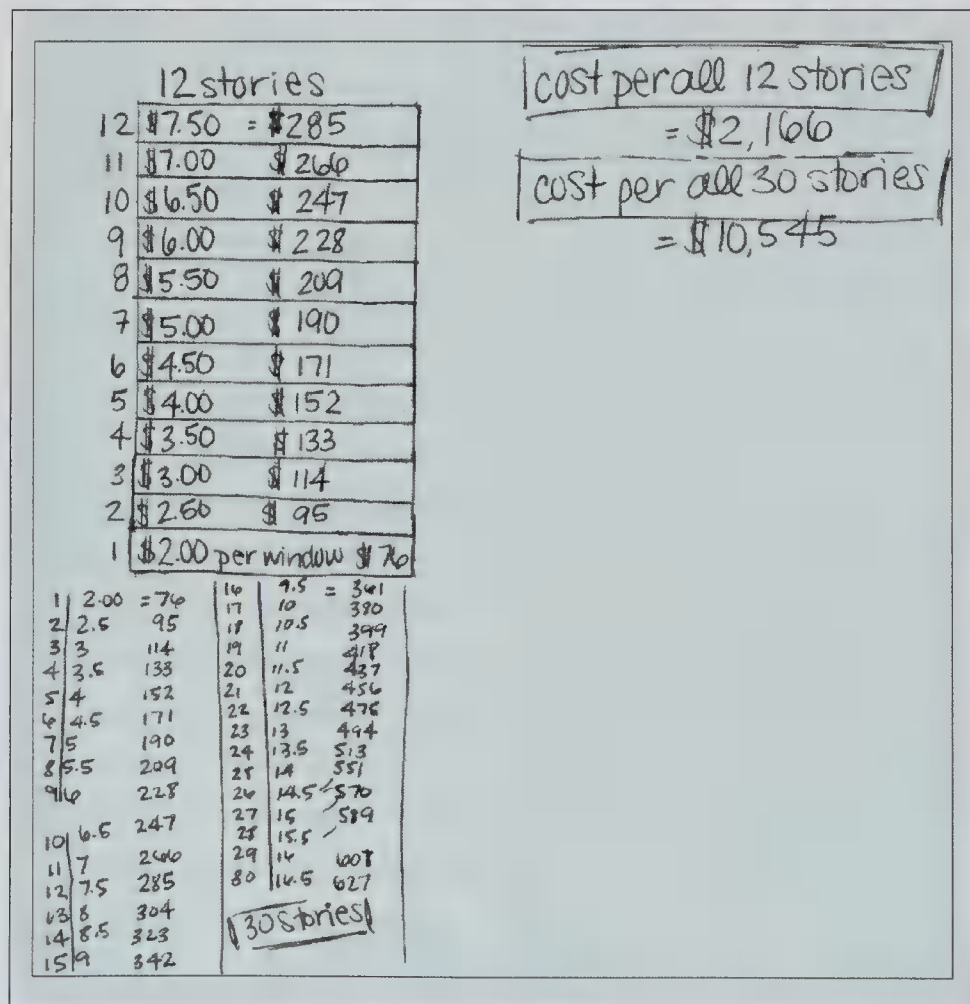


Fig. 1 Students produce a table for calculations.

3. Prepare possible questions and prompts to move students' algebraic thinking forward. Base your questions on what you think might be most challenging to students when they solve this task.
4. Provide time for students individually to think through the mathematics in the problem.
5. Group students to share and develop their individual ideas as well as see how others solve the task.
6. Have groups share their solution strategies with the whole class, starting with less sophisticated strategies and progressing. This approach allows students to build on their own ideas.

As they work on the Skyscraper Windows task, students might take a number of different paths.

LEVELS OF MATHEMATICAL SOPHISTICATION

We focus on six possible student pathways, organized by increasing level of mathematical sophistication. Pathways 1, 2, and 3 demonstrate ways to identify patterns in the data, to calculate the total price using a table, and to find the price for single floors using a formula. Pathways 4, 5, and 6 are possible methods for finding a generalizable solution for the total cost of washing all the windows in a building. Within each pathway are solutions that the preservice teachers shared.

Prompts that teachers might use to help students move along one pathway to the next are included. These prompts are fairly open-ended and support students in thinking about their reasoning, solution paths, and possible next steps. Clearly, those listed are merely some of the myriad possibilities.

Pathway 1: Totaling the Cost within a Table

An initial strategy may be to create a table that organizes the cost of washing windows on each floor, or story (see **fig. 1**). Students using this strategy could fill in the values of a table—up to 12 or 30 floors to calculate the total cost for the buildings.

Filling in a table will not enable students to find a generalizable solution; however, organizing data is a way to aid students in persevering with the problem. With teacher support, students can look for patterns and consider additional steps as they work toward finding a more generalizable solution. Students might also graph the cost of washing the windows on each floor and look for patterns within these graphs.

Possible prompts that a teacher could pose include these:

- Describe patterns or relationships that you notice in your table or graph.
- What is the relationship between the costs of

washing the windows on each floor as the building gets taller?

- Explain the relationship between the number of floors and the cost of washing the windows.

Pathway 2: Finding Patterns in the Table

This pathway, shown in **figure 2**, allows students to see whether any patterns emerge from their tables of values. This strategy builds on pathway 1 in that students would be looking past simply organizing data to seek patterns in the data.

In **figure 2**, the focus is on calculations, not on using variables or finding a more general function. Students can determine the total cost of washing windows per floor, identify how the cost of washing windows changes per floor, and find what is constant and what is changing. Students might also notice that multiples of \$19 are applied to the cost for each floor.

Some prompts that would help students see patterns in their table and look for a generalizable solution follow:

- When you calculate the cost of washing windows for each floor, regardless of which floor, what remains the same, or constant, and why?
- What changes within your table for each floor?
- What patterns do you see between the cost of washing windows per floor and the number of floors?
- Explain which information could help you predict the cost of washing windows for a given floor as well as the total cost.
- Pay attention to aspects in the sequence that might make shortcuts possible for a more general solution (not just for 12 or 30 floors).

Pathway 3: Finding the Cost of Washing Windows for Individual Floors

The table in **figure 2** can help students find a way to write an equation that models the cost of washing the windows for a single floor. **Figure 3** illustrates an expression to determine the cost of washing windows for a specific floor; this student noticed the pattern of a \$19 increase per floor and \$0.50 increase per window per floor.

Two possible equations that students might suggest are these:

- $38(\$2.00 + \$0.50(n - 1))$ = the price for a single floor, where 38 is the number of windows, \$2.00 is the cost per window on the first floor, n is the floor number, and $\$0.50(n - 1)$ is the additional \$0.50 per window on floors above one.
- $\$76 + \$19(n - 1)$ = the price for a single floor, where \$76 is the cost for the first floor, \$19 is the price difference between floors, and n is the floor number.

floor(n)	#/window X	windows =	Total F(n) Price / floor	Total Price = floor
1	\$2	38	$\$76 = 76 + 19(0)$	19
2	\$2.50	38	$\$95 = 76 + 19(1)$	19
3	\$3	38	$\$114 = 76 + 19(2)$	19
4	\$3.50	38	$\$133 = 76 + 19(3)$	19
5	\$4	38	$\$152 = 76 + 19(4)$	19
6	\$4.50	38	$\$171 = 76 + 19(5)$	19
7	\$5	38	$\$190 = 76 + 19(6)$	19
8	\$5.50	38	$\$209 = 76 + 19(7)$	19
9	\$6	38	$\$228 = 76 + 19(8)$	19
10	\$6.50	38	$\$247 = 76 + 19(9)$	19
11	\$7	38	$\$266 = 76 + 19(10)$	19
12	\$7.50	38	$\$285 = 76 + 19(11)$	19
			<u>2166</u>	

Fig. 2 Searching for patterns in the table builds on pathway 1.

What I know:

diff

1st → \$76
2nd → \$95
3rd → \$114
4th → \$133

> 19
> 19
> 19

- add \$0.50 per floor per Window
- let X = Number of floors
- # of Windows remains constant (38 per floor)
- for each additional floor, cost is additional \$19.

linear relationship $1.50 + 0.50n$ = cost for that floor so: $38(1.50 + 0.50n)$

Fig. 3 Students create an equation for a single floor.

Finding a generalizable formula for the price of washing windows on a single floor can provide an entry point to finding the total cost of washing all the windows in the building. A key goal to move student thinking forward is to help students consider what they know about a single floor and how they might use that information to find the total cost.

Some prompts that could move students to think of a generalizable equation for all floors follow:

- Tell which variables you use in your equation and explain why.
- How could you move from calculating the cost of washing the windows on a single floor to calculating the cost for washing all the windows on all the floors? How can you predict the cost of washing windows for all floors without doing the calculations for each individual floor?
- If you have a budget of \$2,500 for window washing, how many floors of windows can be washed? What if you had a budget of \$40,000?
- Look for a pathway to a solution to find the total cost:

- List how you would calculate the cost for washing the windows for the first six floors.
- How might you group numbers according to the patterns you see and why? What operations are you doing consistently each time?
- What numbers are you using repeatedly?
- What steps are you repeating multiple times?

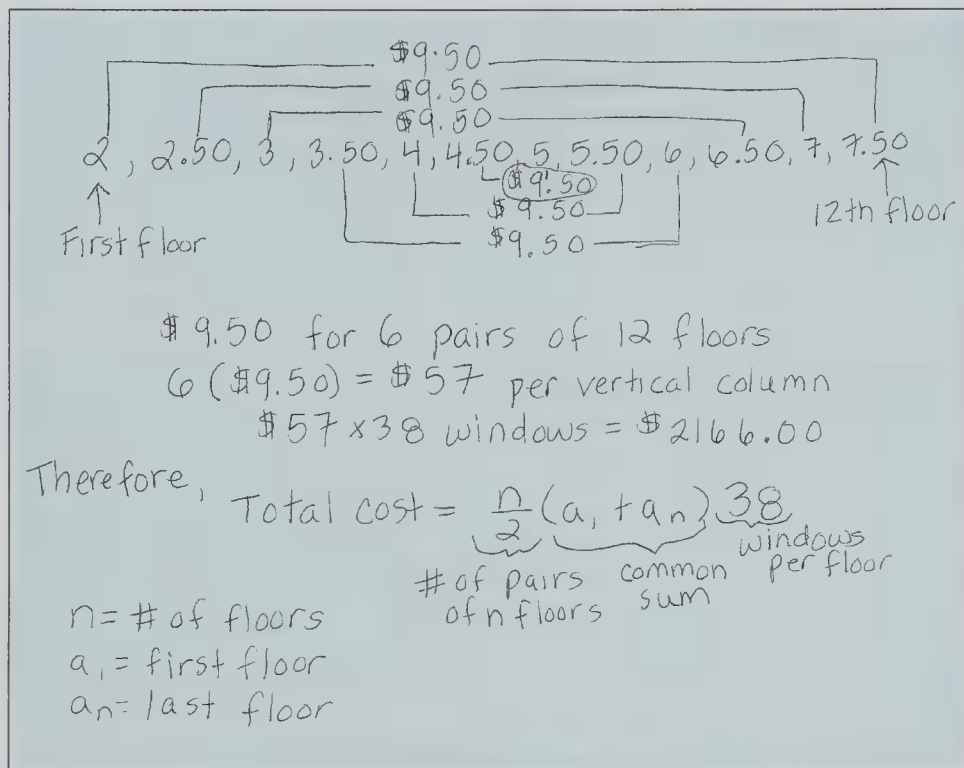


Fig. 4 Using a pairing strategy helps students identify a common sum.

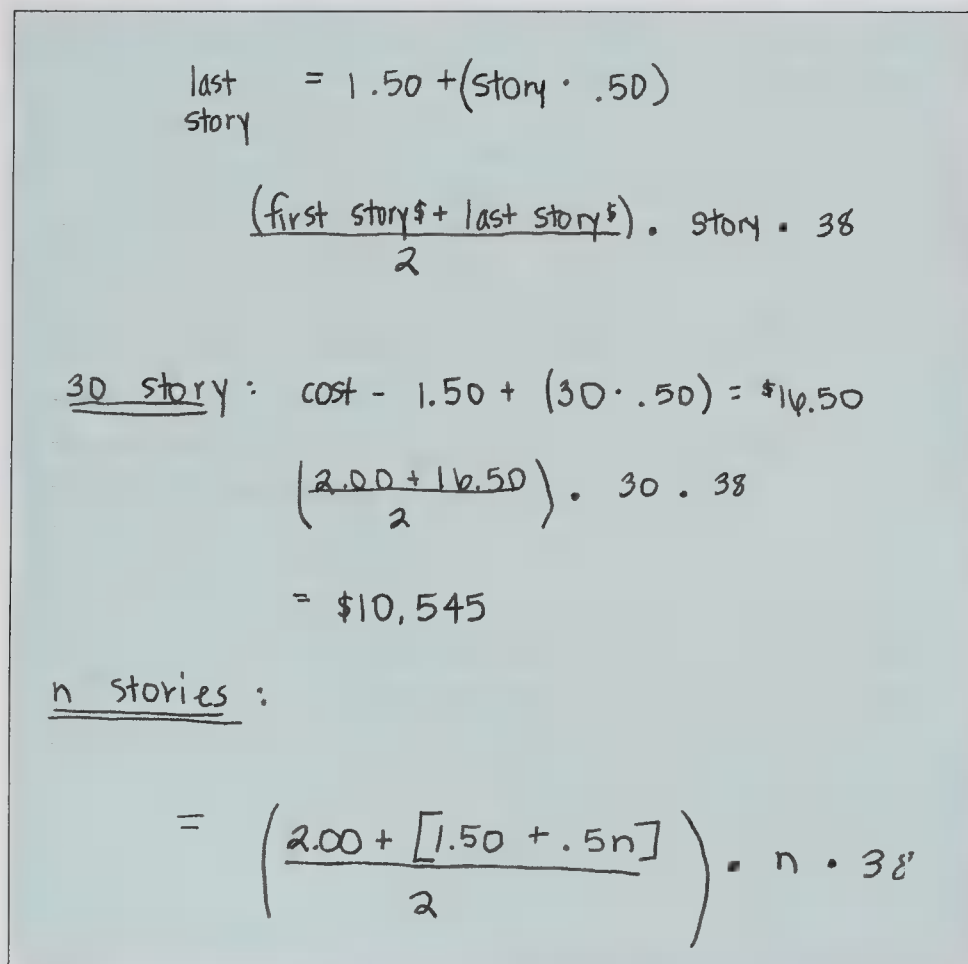


Fig. 5 Using the price of washing the windows on the first and last floors establishes the mean.

The next three pathways provide multiple strategies for finding the total cost of washing all the windows of a building using a generalizable form. Although students could easily use graphing in pathways 1, 2, and 3 (e.g., graphing the total cost for a single floor), graphing is a less likely approach for pathways 4, 5, and 6.

Pathway 4: Using a Pairing Strategy

Figure 4 demonstrates the use of pairing strategies to calculate the total cost of washing all the windows in a building with n floors. This strategy calculates the total cost of washing windows by pairing individual floors to find the cost of each set of paired floors. Students found that if they paired floors, the cost was the same across pairings, allowing them to find the total cost by multiplying the common cost by the total number of floors. **Figure 4** shows how one student has paired the top floor with the bottom floor and then moved incrementally to pair top and bottom floors across a 12-story building.

The mathematical sophistication of this strategy lies in seeing how the number of pairs relates to the number of floors. Students would need to make sense of their strategy by checking conditions, such as whether the pairing strategy works for both even and odd numbers of floors. Students who follow pathway 4 have a developing knowledge of mean. The pairing strategy pushes students from considering how to calculate the cost for a single floor to considering how the costs of different floors relate to one another. To support students in persevering and making sense of their strategy, teachers would likely want their students to think about how the pairing strategy may be connected to the mean.

Several prompts can help students advance their understanding of the pairing strategy:

- How might you be able to group or pair numbers so that you see some patterns? Why might this approach be useful for finding a generalizable solution?
- How might grouping pairs of numbers relate to the number of floors?
- Explain whether your rule works for all cases. Does it matter whether the number of floors is even or odd? Why or why not?
- How does grouping pairs of numbers to find the common sum help you understand how the mean may be involved?
- What does each term in your equation represent? How might these terms help you understand the mean?

Pathway 5: Using the Arithmetic Mean

Because the increase in cost per floor is steady, we can calculate the total cost of washing all the

windows on all the floors by finding the average (mean) cost per floor. **Figure 5** demonstrates how one student has found the mean price of the last and first stories, thus allowing the student to multiply the mean by the number of floors and the number of windows to arrive at the total cost. In this example, the student did not need to work through all the floors, as was done in **figure 4**, and has instead extrapolated the pattern across all floors.

This method is somewhat more sophisticated than the previous pathways because it highlights the relationship between the total costs for floors and the total number of floors. Students who use this strategy may have the conceptual understanding of using the mean; they realize that they do not have to calculate the sum of all the floors and divide by the total number of floors.

To support students in persevering and making sense of their mathematics, teachers would likely want their students to think about how and why the mean helps determine a total price. Some prompts that would help students use this strategy follow:

- What is the average (mean) cost of each floor? How might knowing the mean help you construct a generalizable equation to calculate the total cost of washing the windows for all floors?

- What does each term in your equation represent? How do the numbers in your equation relate to the problem's context?
- Explain whether your rule works for all cases. Does it matter whether the number of floors is even or odd? Why or why not?

Pathway 6: Using Arithmetic Sequence and Series

The sixth pathway may not be feasible for beginning algebra students but is certainly one that advanced algebra students might identify. Students need to recognize the data as a sequence and use their understanding of how the cost of washing windows on one floor influences the cost for the next floor to obtain a general form. The student whose work is shown in **figure 6** understands that a pattern exists that forms a sum and then uses that relationship to find the answer and general form.

Using sequences and summations builds on prior ideas about identifying patterns. To support students in persevering in this method, teachers should make sure that students understand each part of the equations and expressions they are using and how they connect to previous work the class has done on this topic.

New from NCTM: The Essential Guide to Navigating Your First Years of Teaching Secondary Mathematics

FUTURE. | INSPIRING TEACHERS. ENGAGING STUDENTS. BUILDING THE FUTURE. | INSPIRING

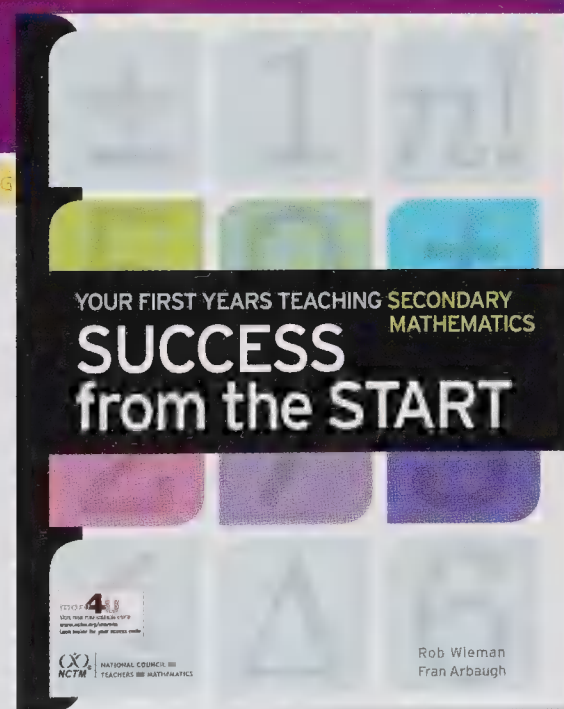
Success from the Start: Your First Years Teaching Secondary Mathematics

BY ROB WIEMAN AND FRAN ARBAUGH

You just signed your first contract to teach secondary math. You're excited but you have many questions and concerns:

- What do I do when students don't "get" the lesson?
- What about students who struggle with math they supposedly learned in elementary school?
- How do absent students make up the work?
- Do I assign seating or let students sit wherever they want?
- Should I let students work in groups?
- How much homework should I assign and grade?

Based on classroom observations and interviews with seasoned and beginning teachers, *Success from the Start: Your First Years Teaching Secondary Mathematics* offers valuable suggestions to improve your teaching and your students' opportunities to learn. The authors explore both the visible and invisible aspects of teaching and offer proven strategies to make the work meaningful—not merely manageable. Success from the start means being prepared from the start. This book not only teaches you how to be an effective math teacher but also gives you the tools to do it well.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

SAVE 25% on this and ALL NCTM publications. Use code **MT813** when placing order. Offer expires 9/30/13. *This offer reflects an additional 5% savings off list price, in addition to your regular 20% member discount. For more information or to place an order, please call (800) 235-7566 or visit www.nctm.org/catalog.

$$\begin{aligned}
\text{Total Cost} &= \sum_{n=1}^k 38(1.5 + .5n), \quad k = \# \text{ of floors} \\
&= 38 \left[\sum_{n=1}^k 1.5 + \sum_{n=1}^k .5n \right] \\
&= 38 \left[1.5k + \frac{.5k(k+1)}{2} \right] \quad \text{b/c } \sum_{n=1}^k n = \frac{k(k+1)}{2} \\
&= 38(1.5k) + 38 \left(\frac{.5k(k+1)}{2} \right) \\
&= 57k + \frac{19k(k+1)}{2} \\
&= \frac{114k + 19k(k+1)}{2} \\
&= \frac{114k + 19k^2 + 19k}{2} \\
&= \frac{19k^2 + 133k}{2} \\
&= \frac{19k(k+7)}{2}
\end{aligned}$$

Fig. 6 The strategy of using sequences and series will not occur to most beginning algebra students.

Some prompts to push students' thinking related to finding an arithmetic sequence and series include these:

- Define your terms. What does each term in your equation represent? How do the numbers in your equation relate to the problem's context?
- How does a summation take into account the total cost of washing the windows on all floors?
- What are other ways to write your expression? Explain what your new method tells you that a previous one did not.
- Pay particular attention to your indices. What do they represent?

PUSHING STUDENTS' THINKING FORWARD

The Skyscraper Windows task is a cognitively demanding algebra task that teaches to the Common Core and connects to the mathematical practice of making sense of problems and persevering in solving them. Such tasks allow students to deconstruct and solidify their understanding of algebraic concepts and see the multiple pathways possible within a single problem. This task also allows teachers to see how a single problem can be used for multiple purposes across different levels and grades of algebra instruction, depending on instructional goals. Further, tasks such as this one give students a chance to solve problems algebraically; they support students in building rules to represent functions and abstracting from computation, two of Driscoll's (1999) algebraic habits of mind.

Tasks that elicit problem-solving experiences, such as Skyscraper Windows, allow students and teachers to engage with mathematics in a truly meaningful way. However, implementing such tasks takes work on the part of both students and teachers. Teachers must identify goals for a task, prepare for possible student solutions, and consider how to respond to students' work and thinking without diminishing the cognitive demand of the task. By examining and imagining how to use a cognitively demanding task, teachers can prepare questioning strategies to move students' algebraic thinking forward and to engage students in meaningful mathematics experiences that align with the Common Core.

REFERENCES

- Boston, Melissa D., and Margaret S. Smith. 2009. "Transforming Secondary Mathematics Teaching: Increasing the Cognitive Demands of Instructional Tasks Used in Teachers' Classrooms." *Journal for Research in Mathematics Education* 40 (2): 119–56.
- Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf
- Driscoll, Mark. 1999. *Fostering Algebraic Thinking: A Guide for Teachers, Grades 6–10*. Portsmouth, NH: Heinemann.
- Smith, Margaret S., and Mary Kay Stein. 2011. *Five Practices for Orchestrating Productive Mathematics Discussions*. Reston, VA: National Council of Teachers of Mathematics.
- Stein, Mary Kay, Barbara W. Grover, and Marjorie Henningsen. 1996. "Building Student Capacity for Mathematical Thinking and Reasoning: An Analysis of Mathematical Tasks Used in Reform Classrooms." *American Educational Research Journal* 33 (2): 455–88.



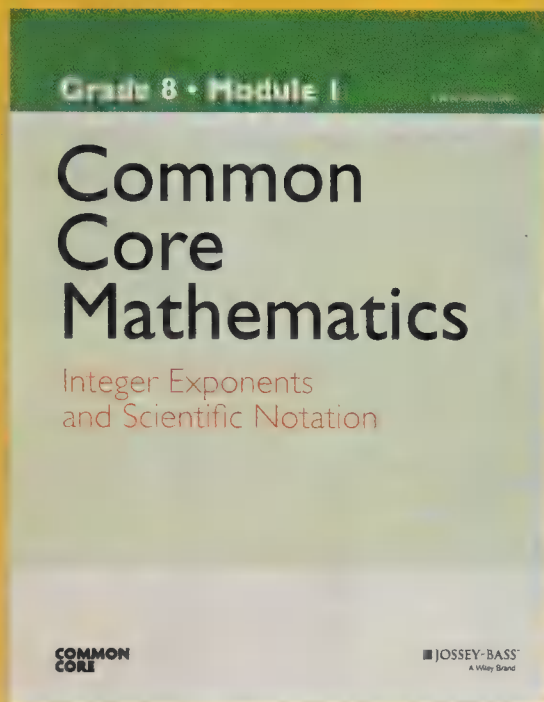
SARAH A. ROBERTS, sroberts@iastate.edu, is an assistant professor of mathematics education at Iowa State University in Ames. Her interests include equity in mathematics education, supporting English language learners in mathematics, and professional development for pre- and in-service teachers.



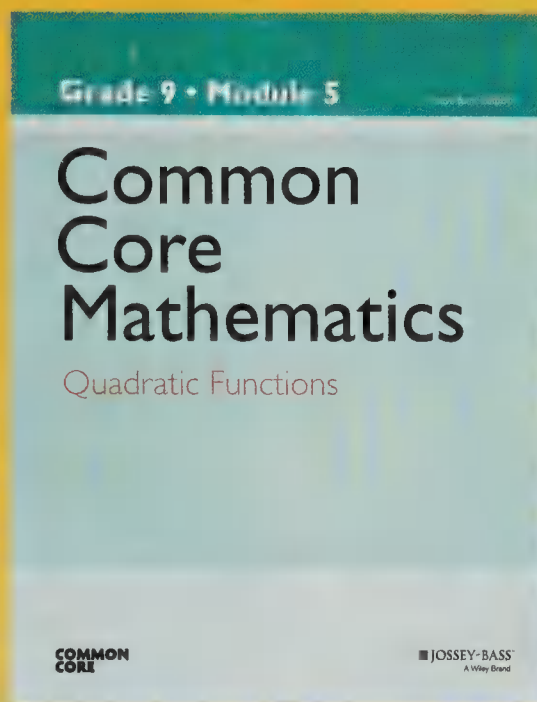
JEAN S. LEE, jslee@uindy.edu, is an assistant professor of teacher education at the University of Indianapolis. Her interests include project- and problem-based learning and preparation of teachers for high-need, urban school settings.

Common Core Mathematics

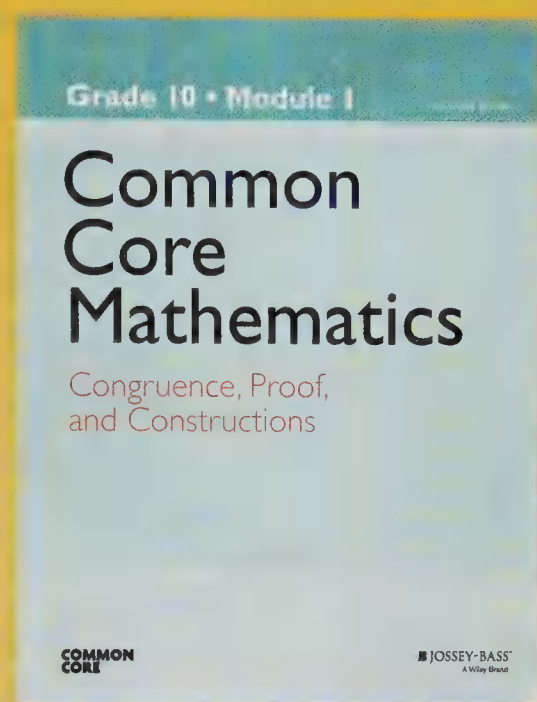
available in print from Jossey-Bass



978-1-118-79370-1



978-1-118-81128-3



978-1-118-79368-8

Common Core Mathematics, New York State Edition is a carefully sequenced, content-rich curriculum featuring 180 days of instruction across approximately six units (modules) per grade (K-12).

Developed by Common.org and the New York State Education Department and targeted for attainment of CCSS by all learners within any classroom setting, this curriculum incorporates:

- year-long scope and sequence documents
- lesson plans
- performance tasks
- scaffolding materials
- samples of student work
- formative assessments
- student worksheets

Appropriate for K-12 classes in schools across the U.S. and previously accessible only online, the curriculum will now be available in **convenient and inexpensive, full color printed and bound books**, one for each module by grade. They are being published throughout the 2013-2014 school year. Order now to get the complete set for any grades.

Complete ordering information, discount options, sample chapters, and publication schedule are all available at www.wiley.com/go/ccss/jossey.

JOSSEY-BASS
A Wiley Brand



Cra
&
R

Cracking Codes Launching Lockets

This historically significant real-life application of a cryptographic coding technique, which incorporates first-year algebra and geometry, makes mathematics come alive in the classroom.

Teo J. Paoletti

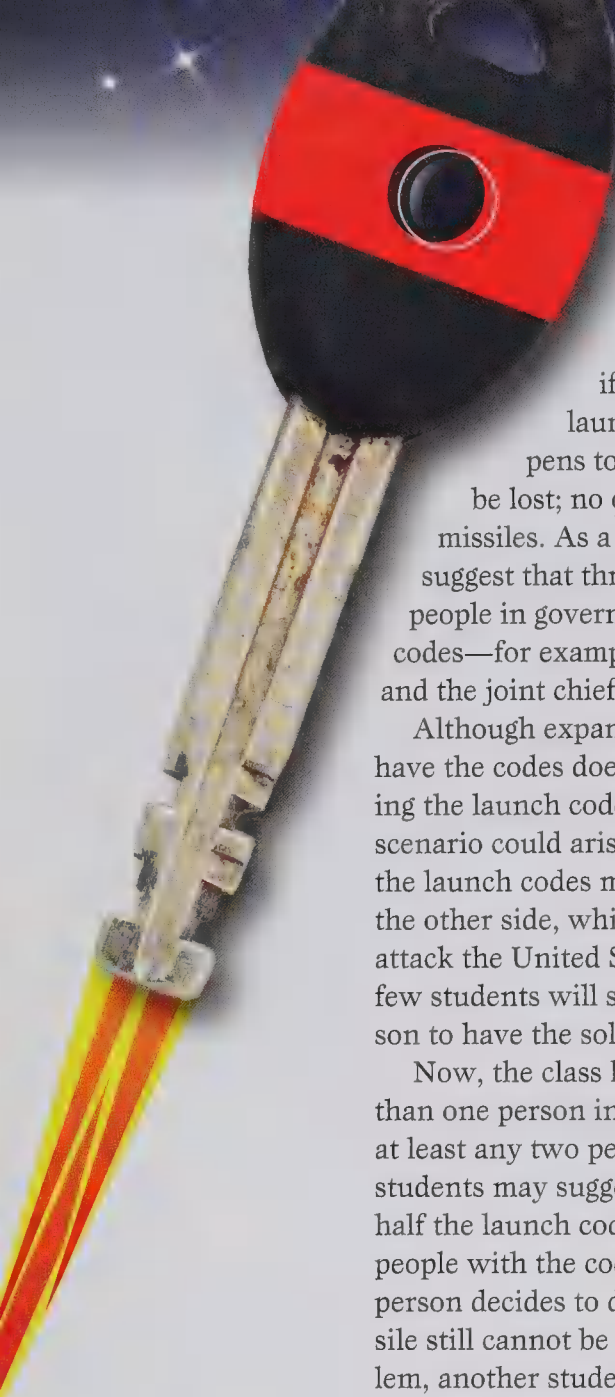
To engage students, many teachers wish to connect the mathematics they are teaching to other branches of mathematics or to real-world applications.

The lesson presented here, which uses the algebraic skill of finding the equation of a line between two points and the geometric axiom that any two points define a line, does both. A historically significant coding technique relies on algebra skill and the axiom (Trappe and Washington 2006). Further, this lesson integrates the first Standard for Mathematical Practice from the Common Core State Standards (CCSSI 2010) and many of the NCTM Process Standards—namely, Problem Solving, Reasoning and Proof, Communication, and Connections (NCTM 2000). What follows is a description of how the lesson can be presented as well as insight into adaptations that will make it a good fit for any classroom.

WHO SHOULD HAVE THE CODE?

After World War II, tensions between the United States and the Soviet Union ran high. The period known as the Cold War saw a proliferation of nuclear weapons by both powers (Schroeer 1984). Luckily, both were concerned with the question of who should and who should not have the ability to launch these destructive weapons. This question remains relevant today, considering the development of Iran's nuclear program and the threat it poses to stability in the Middle East.

After giving some brief historical background, I ask students to imagine that they are in charge of distributing the launch codes for the United States. They then have a few minutes to think about how they would handle the situation. During the ensuing class discussion, students find many legitimate issues when it comes to protecting the launch codes.



Many students' first thought is to give only the U.S. president the ability to launch the missiles. However, during the conversation, students come to realize that if only the president is given the launch codes and something happens to him or her, then the codes will be lost; no one else will be able to launch the missiles. As a result, some students typically suggest that three or four of the most important people in government should be given the launch codes—for example, the president, vice president, and the joint chiefs of staff, among others.

Although expanding the number of people who have the codes does decrease the possibility of losing the launch codes altogether, a more disastrous scenario could arise. One of the many people with the launch codes might defect to or be captured by the other side, which could use these missiles to attack the United States. At this point, typically a few students will state that it is unwise for one person to have the sole ability to launch a missile.

Now, the class has to decide how to give more than one person information that would require at least any two people to launch a missile. Some students may suggest giving each of the two people half the launch code. However, having only two people with the code still creates problems. If one person decides to defect, is captured, or dies, a missile still cannot be launched. To resolve this problem, another student may suggest that there should be four people, two with half the launch code and two with the other half. Although this suggestion addresses the issue of being unable to launch the code if only one person defects, it still requires one person from each group to be present to launch the missiles, which is less than desirable because the goal is for any two of the important people to be able to launch the missile together.

Clearly, as the class discusses how to divide the launch code, the problem proves to be much more complicated than students may have originally thought. At this point, the problem of protecting the launch codes may not have been resolved. However, through the brief classroom conversation, the students are doing some valuable work. The situation leads them to solve a problem in a nonmathematical context and communicate their solution strategies to their peers and teacher, thus addressing NCTM's Problem Solving and Communication Standards (NCTM 2000). Further, as the discussion motivates the class to devise improved strategies, students persevere in problem solving, which is the first Standard for Mathematical Practice in the Common Core State Standards (CCSSI 2010).

I typically devote eight to ten minutes to this initial brainstorming discussion. Some students may sit silently for the first two minutes, but, by the latter part of the conversation, most are engaged with the task. At this point, the class summarizes its ideal system. The goal is to create a system in which no one individual has enough information to launch the missile. Further, more than just two people should possess enough information so that, if any two of them get together, those two have the ability to launch the missile.

ALGEBRA, GEOMETRY, AND THE TWO-PERSON RULE

The first key idea needed is the geometric axiom that any two points determine a line. Although almost all the students will already understand this idea, it is critical to this application. In introducing the axioms in a geometry class, teachers could use this lesson as a way to engage students with a historically significant, real-life problem whose solution relies on an axiom.

If several people are given coordinates of a point on the same line, none will be able to find an equation of the line with only his or her coordinates. However, any two of these people will have the ability to determine the equation of the line that extends between their two points. It does not matter which two people share their knowledge; all the points are on the same line. Thus, we create a framework in which one person does not have enough information to do anything, but any two persons can share their knowledge to construct something useful. This technique allows the class to choose multiple high-ranking officials such that any two can join forces to launch the missiles.

Now that students have an idea of how to design a system whereby a number (the launch code) can be hidden, they still need a way to implement this system. This is where algebra comes in. Each person is given the coordinates of a point in the Cartesian plane. Once two of the people meet, they can use their points to find a slope and then use one of the points and the slope to find the equation of the line. The line has already been designed so that the y -intercept is the launch code. In this ingenious design, students make connections between mathematical ideas, an element of NCTM's Connections Standard (NCTM 2000).

I then introduce the term *two-person rule*, which is the proper term for the cryptographic procedure (Trappe and Washington 2006). The use of the two-person rule can be seen in movies such as *The Sum of All Fears*, *The Hunt for Red October*, and *WarGames*. If a teacher has access to these movies, presenting the appropriate clip will help engage students.

Column A	Column B	Column C	Column D	Column E
(5, 1348)	(2, 1621)	(-5, 1877)	(-6, 1769)	(19, 234)
(3, 1336)	(12, 1681)	(13, 1985)	(-7, 1763)	(-11, 54)
(11, 1384)	(-8, 1561)	(15, 1997)	(8, 1853)	(-15, 30)
(-4, 1294)	(-2, 1597)	(6, 1943)	(14, 1889)	(-13, 42)
(-9, 1264)	(1, 1615)	(4, 1931)	(4, 1829)	(16, 216)
(-1, 1312)	(7, 1651)	(-3, 1889)	(10, 1865)	(18, 228)

Fig. 1 First-year algebra can be used to break the code and launch the rocket.

Suppose that the launch code is 1234. I ask a student to pick a slope from 5 to 15. The most common student choice of slope is 7. Using this number, I write the equation $y = 7x + 1234$ on the board, stating that all the points will come from this line. I then pick five x -values and find the corresponding y -values. For instance, I write the following five points on the board: (5, 1269), (-3, 1213), (-7, 1185), (8, 1290), and (7, 1283). Then I pick five volunteers to be the people who have some information about the launch code, and I give each only one of the points. I then erase the original equation.

Students then discuss whether or not any of the five people can find the equation of the line on his or her own. Students typically see that none of them, individually, has enough information. Next, the class decides which two students should launch the missile, and those two work on the board to find their slope and the resulting equation. Their work produces the previously erased equation, $y = 7x + 1234$, leading to the chosen launch code of 1234.

At this point, some students will object, claiming that whoever gives out all the points already knows the launch code. However, to prevent this situation in real life, a computer would randomly generate both the launch code and the slope. The computer would store the launch code in the appropriate place and print out five points, so that no one person knows the code.

THE LAUNCH CODE ACTIVITY

After about fifteen to eighteen minutes with the activity above, students are typically intrigued by and engaged with the lesson. A next exercise can be to give each student a point and a partner and have the two of them find a launch code; this exercise would be somewhat uninteresting but could be used if time is limited. A better exercise is to give each student the coordinates of a point and tell students that they each have a launch partner in the class. They will know that they have found their partner when the slope between their two points is an integer between 1 and 10.

What the students do not know is that the coordinates are designed so that there are only five unique lines, one per column, each with a slope of 6 (see **fig. 1**). As a result, each student may have more than one launch partner, depending on class size. For example, with twenty-four students, the teacher could distribute six pairs from each of the first two columns of the chart in **figure 1** and four pairs from each of the last three columns, giving a total of twenty-four points. However, once any two of the six people from the first column find each other, they will be launch partners.

If the number of students is odd, the teacher can become the extra person but have the students do all the calculations to check whether they are launch partners. As students begin to find partners, other students have fewer people to check with; eventually, almost everyone finds a launch partner. Usually, almost all twenty-four students will find their launch partner within ten to fifteen minutes.

The choice for the slope of the integer value 6 enables quicker location of a launch partner. With multiple slopes of different values, there is a slight chance that a pair of students will find one of the possible correct slopes using two points from different lines. Because of this possibility, I use the same integer slope for each line.

Although negative slopes can be used, I recommend against using fractional slopes. Students may have problems with rounding, causing difficulty in finding an exact launch code. Further, the real-life application of the two-person rule uses integers (typically large primes) and modular arithmetic. Hence, the application itself uses all integers.

One by-product of this lesson is the practice students get in finding the slope between two points. Students typically work with at least two and often many more students to find the slope between two points as they try to find their launch partners, repeating the process as many times as required to find the integer between 1 and 10. Further, after

trying multiple points, some students develop some intuition. Knowing that they are looking for a slope between 1 and 10, they can often look at another student's point and decide quickly whether it will work or not. One student will often be overheard explaining to another that the difference in their x -values will be a small number but that the difference in their y -values will be a large number; as a result, the ratio of the differences of the y -values to the x -values cannot be an integer between 1 and 10.

Such interaction—as students apply the slope formula while using their number sense to estimate a ratio and make a conclusion based on all this information—is always one of the best parts of the lesson and is highly representative of NCTM's focus on reasoning and sense making (NCTM 2000). Further, as two students explain their thought processes, other students understand why they are able to deduce quickly that the pair are not launch partners. This understanding connects with NCTM's Communication Standard as well (NCTM 2000). Students construct viable arguments and critique the reasoning of others as well as attend to precision—the third and sixth Standards of Mathematical Practice of the Common Core State Standards (CCSSI 2010).

Once a student finds his or her launch partner, the two must find the equation of the line. After they have found the equation, they must report their launch code (y -intercept) to the teacher, who will verify that it is correct. If it is incorrect, the students are asked for the slope that they found. If their slope is 6, they are asked to find their equation again more carefully. If the slope is not 6, it will typically be a noninteger between 1 and 10 or an integer greater than 10. Thus, students' misunderstanding about what the slope is can be clarified, and the pair can be asked to keep looking for their launch partners.

EXTENSION: CONVERTING LETTERS TO NUMBERS

Students who find their launch partners early can be informed that this problem has another level. In cryptography, letters are often replaced by their numerical value ($A = 01$, $B = 02$, ..., $Z = 26$). Ask students to convert their launch code to a two-letter combination. From the coordinates in **figure 1**, the five launch codes are 1318 (MR), 1609 (PI), 1907 (SG), 1805 (RE), and 120, or 0120 (AT), which spells MR P IS GREAT. A teacher can adjust the message by choosing a saying, converting the letters to numbers, and making these the y -coordinate of the y -intercepts of different lines.

For instance, if a teacher would rather use the message MATH IS COOL, he or she could convert two-letter combinations ($MA = 1301$, $TH = 2008$, $IS = 0919$, $CO = 0315$, $OL = 1512$) and use these numbers as the y -intercepts of different lines, all with the same slope. Then the teacher would simply pick unique x -values, finding y -values from the different lines, until he or she has enough points for everyone in a class. This exercise adds another level of intrigue and interest that can be explored, especially if all students finish early.

A LESSON FOR EVERYONE

Students find this lesson both interesting and engaging. They enjoy the historical aspect of the Cold War. However, some teachers do not like the idea of integrating a war scenario into their mathematics class. Luckily, the two-person rule has other applications, including banking and IT security. However, the war scenario is historically accurate, giving credence to the applicability of the mathematics. Further, students are less interested in and less knowledgeable about banking or IT security than they are about World War II and the Cold War.

Students are intrigued by the idea of cracking codes. In this activity, disaffected students engage with classmates in an attempt to find a launch code. I have even had students who, months after the class, ask, "Can we find launch codes again today?" Connecting history, geometry, algebra, and cryptography makes this an interesting and enjoyable lesson for all students.

REFERENCES

- Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf
- National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- Schroeer, Dietrich. 1984. *Science, Technology, and the Nuclear Arms Race*. New York: Wiley.
- Trappe, Wade, and Lawrence C. Washington. 2006. *Introduction to Cryptography with Coding Theory*. 2nd ed. Upper Saddle River, NJ: Pearson Prentice Hall.



TEO J. PAOLETTI, paolett2@uga.edu, taught mathematics at Moorestown High School in Moorestown, New Jersey, before pursuing his PhD in mathematics education at the University of Georgia in Athens. He is interested in student learning, curriculum development, and secondary and postsecondary school mathematics.

2015 Focus Issue

Creating Classroom Communities



NCTM guides mathematics teachers toward equitable teaching by emphasizing the importance of classroom communities. The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, and education researchers to share their experiences in building classroom communities. We encourage submissions that will help *MT* readers learn new ways to capitalize on the strengths that cultural, racial, and linguistic diversity bring to our classrooms and schools.

Classroom communities embrace individuals.

- How can we foster a sense of belonging in our classrooms?
- How can we learn about our students—their interests, issues that might be important to them, their languages, and their racial or ethnic communities?
- How can we incorporate this knowledge in our lessons and assessments? What are examples of effective tasks that highlight strengths of individual students? How can we balance individual, cooperative, and whole-class activities?

Classroom communities foster communication.

- How do we organize lessons so that students can share their mathematical ideas or solutions?
- What classroom norms are effective in facilitating communication?
- How can we encourage students to listen to, critique, and build on other students' mathematical thinking?
- How can we communicate high standards for students?
- How can technologies foster student communication about mathematics?
- What strategies are successful in removing barriers to student participation and engagement in mathematics?

Please submit manuscripts at mt.msubmit.net by **May 1, 2014**. Be sure to enter the call's title (Creating Classroom Communities) in the Department/Calls field. No author identification should appear in the text of the manuscript. See www.nctm.org/publications/content.aspx?id=22602 for manuscript guidelines. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.

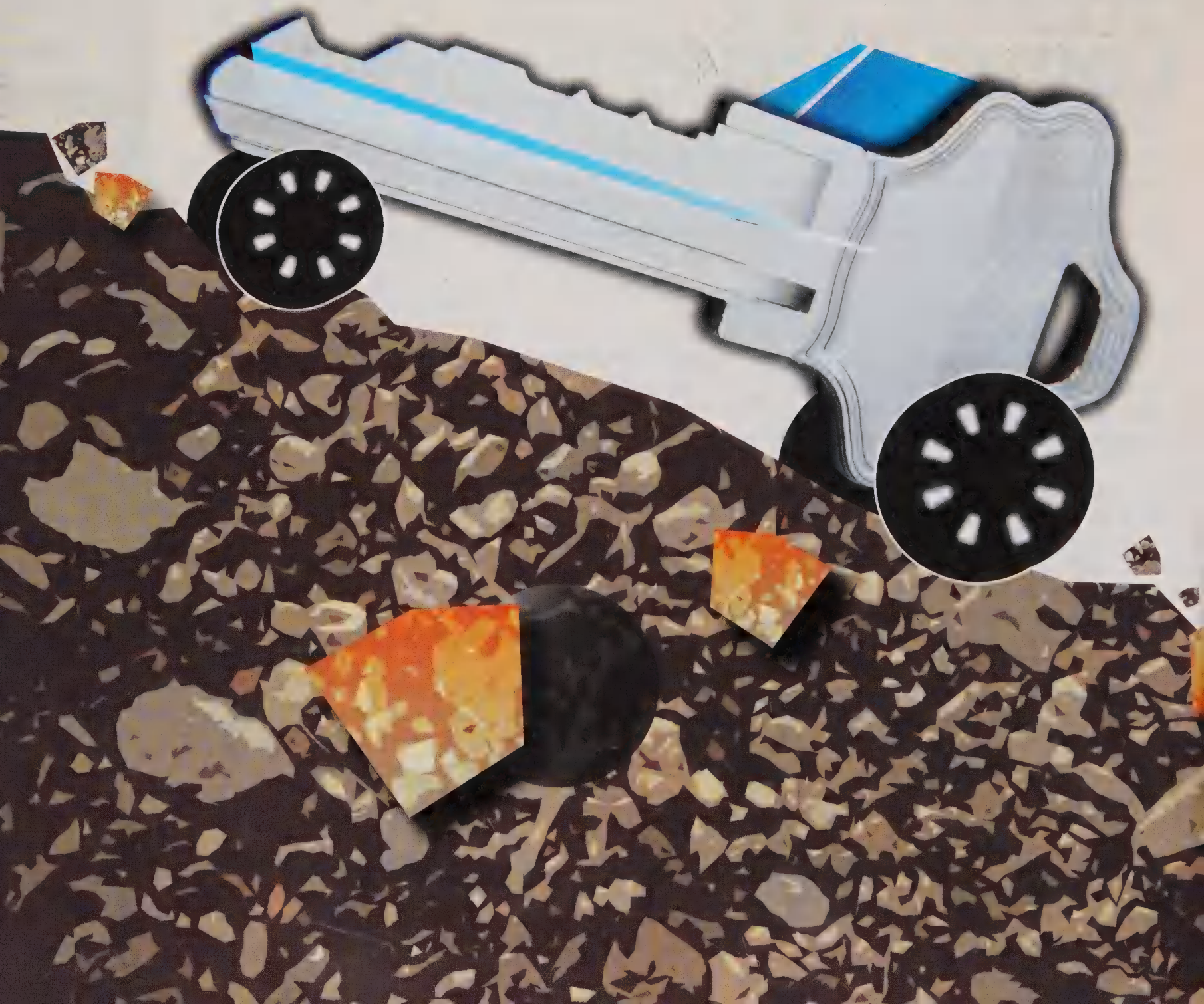


NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

MATHEMATICS
teacher

CALL FOR MANUSCRIPTS

Connecting Steep



Slope, mess, and Angles

Students must be able to relate many representations of slope to form an integrated understanding of the concept.

Courtney R. Nagle and Deborah Moore-Russo

All teachers, especially high school teachers, face the challenge of ensuring that students have opportunities to relate and connect the various representations and notions of mathematics concepts developed over the course of the pre-K–12 mathematics curriculum. NCTM's (2000) Representation Standard emphasizes the importance of students being able to represent a concept in various forms—for example, verbally, numerically, graphically, and analytically—but students' recognition that each representation depicts the same underlying concept is of equal importance. When given one ordered pair and a constant rate of change between two variables, students should be able to graph a line, identify the parameters of a linear equation, and complete a table.

However, being able to use each representation independently is not sufficient; students need to connect the representations and recognize how each indicates a specific relationship between two simultaneously varying quantities. Thus, when teaching slope, the emphasis should be not only on using multiple representations but also on connecting various representations to form a coherent and complete conception of this mathematical idea.

SLOPE ACROSS THE CURRICULUM

Although relating various representations of a concept may be difficult for students, doing so is even more difficult when alternative notions are introduced at different grade levels across the curriculum. This is the case with slope, which is first introduced early in algebra but which continues to be prominent throughout the secondary school

mathematics curriculum, culminating with the derivative in calculus. Ask students to explain slope, and they may offer responses such as these: “rise over run,” “change in y over change in x ,” “indicator of the increasing or decreasing behavior of a line,” “ m ,” “the derivative,” or “the rate of change of a function” (see Moore-Russo, Conner, and Rugg 2011; Stump 1999, 2001). Because various interpretations of slope are emphasized at various stages, it is critical for teachers to provide opportunities for students to make connections with prior knowledge to form an integrated understanding of the concept.

NCTM calls for a curricular focus on sense making, described as “developing understanding of a situation, context, or concept by connecting it with existing knowledge” (NCTM 2009, p. 4). However, relating the various conceptualizations introduced across the mathematics curriculum can be difficult. Further, students’ prior knowledge of some mathematical concepts may also include ideas from everyday experiences that are not explicitly addressed in the curriculum. These informal experiences continue to shape students’ mathematical views of the concept even after formal mathematical definitions are introduced (Tall and Vinner 1981).



Fig. 1 Students gain everyday experience with the concept of slope from highway signs warning drivers about steep gradients.

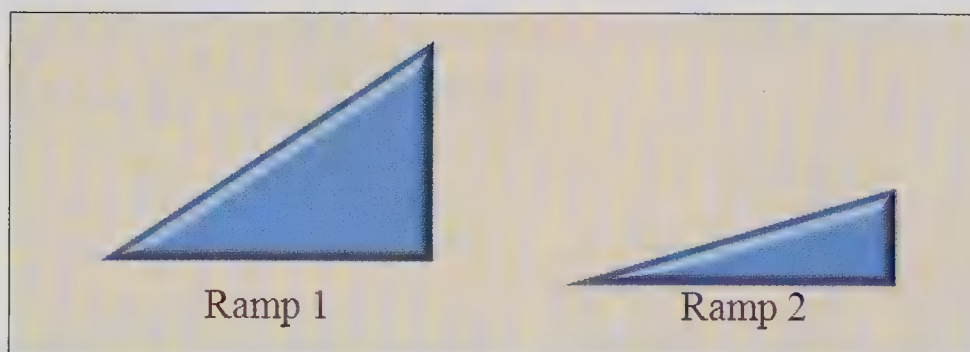


Fig. 2 Students often describe ramp 1 as being steeper than ramp 2 because the angle formed between the ramp and the ground is greater for ramp 1.

Students must be the ones to make sense of concepts, but teachers can provide learning environments that encourage them to build on prior knowledge. When students actively construct new notions from prior knowledge, the new conceptualizations become an integrated part of their complete concept image. Thus, teachers should recognize students’ past experiences with mathematical ideas and consider how this prior knowledge can be used effectively to develop formal mathematical notions.

SLOPE, STEEPNESS, AND ANGLES OF INCLINATION

Before hearing the term *slope* used in a mathematical sense, students gain everyday, nonmathematical experiences of the idea of slope through encounters with steepness and inclines. For instance, children learn that sledding down a steep hill may be fun but walking back up that same steep hill requires a great deal of effort. Other experiences, such as setting the level of incline on a treadmill or reading roadway signs that warn about steep grades (see **fig. 1**), provide valuable opportunities for students to explore the notion of slope.

However, despite the natural relationship between the idea of slope of a line and everyday uses of the terms *steepness* and *incline*, the secondary school curriculum infrequently acknowledges slope as a measure of steepness or inclination. In a study of the fifty states’ standards for teaching mathematics, only five states included slope as a measure of steepness or incline in their standards (Stanton and Moore-Russo 2012). By contrast, standards documents tend to emphasize views of slope as the ratio of rise over run; as an indicator of the increasing, decreasing, or constant behavior of a line; as determining whether two lines are parallel or perpendicular; as indicating the rate of change of a function; and as denoted by $(y_2 - y_1)/(x_2 - x_1)$. Each of these views is an important interpretation of slope, but omitting its more physical interpretation as steepness may prevent students from linking the mathematical ideas with their everyday experiences.

The trigonometric concept, which equates slope with the tangent of the angle of elevation between a line and the x -axis, is also infrequently found in the states’ standards documents (Stanton and Moore-Russo 2012). This is somewhat counterintuitive since a previous study found that students had a natural tendency to describe steepness in terms of angles (Stump 2001). For instance, students might characterize ramp 1 in **figure 2** as steeper than ramp 2 by describing it as forming a greater angle with the ground. This same study showed that students did not relate the idea of steepness to the ratio used to measure the slope of a line, indicating that they viewed steepness as a property of physical

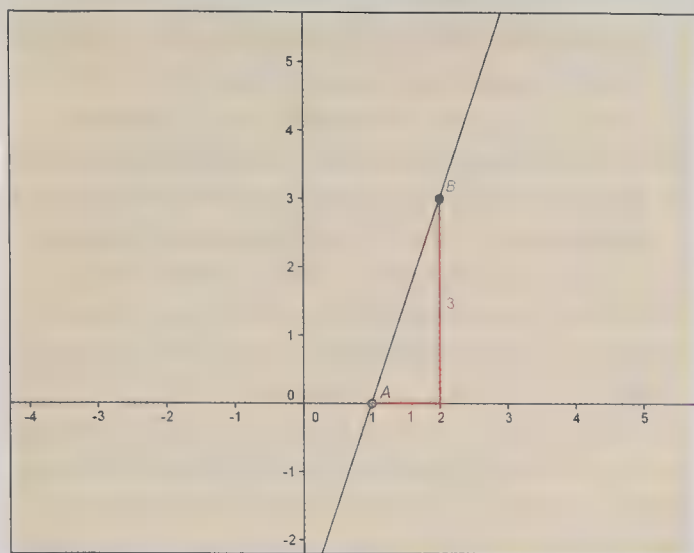


Fig. 3 The given line has a slope of 3, indicated by a slope triangle with a height of 3 units and a base of 1 unit.

objects but isolated it from the mathematical notion of “rise over run” of a linear graph.

Because Stump’s (2001) study suggests that students do relate steepness and angles of elevation, building students’ views of slope as related to angle of elevation may broaden their conceptualizations and also provide critical sense-making opportunities.

THE ACTIVITY

The student activity sheet in the appendix (see pp. 278–79) provides a series of tasks that can be used to engage students in this sense-making process by using GeoGebra or other similar dynamic geometry software. The activities assume that students have prior knowledge of slope as rise over run and as indicating the behavior of a line—whether it increases, decreases, or remains constant. The activities also assume that students are familiar with slope triangles, which are “triangles created by a vertical ‘rise’ line segment (showing the change in y), a horizontal ‘run’ line segment (showing the change in x), and a segment of the line itself” (NCTM 2006, p. 20). GeoGebra emphasizes the geometric interpretation of rise over run by illustrating a slope triangle for a selected line when the slope tool (under the **measurement tools** menu) is used (see **fig. 3**).

These activities encourage students to apply their prior knowledge of slope and steepness as they reason about new problem situations. Such activities give students a chance to make connections between the two ideas and, ultimately, to come to a deeper understanding of slope (Pirie and Martin 2000).

Part 1: Relating Angles and the Measure of a Line’s Slope

Students first explore what happens as the line’s run (the base of the slope triangle) is increased while the line’s rise (the height of the slope triangle) remains constant (see part 1, questions 1–4). Their prior knowledge of slope as the rise-to-run

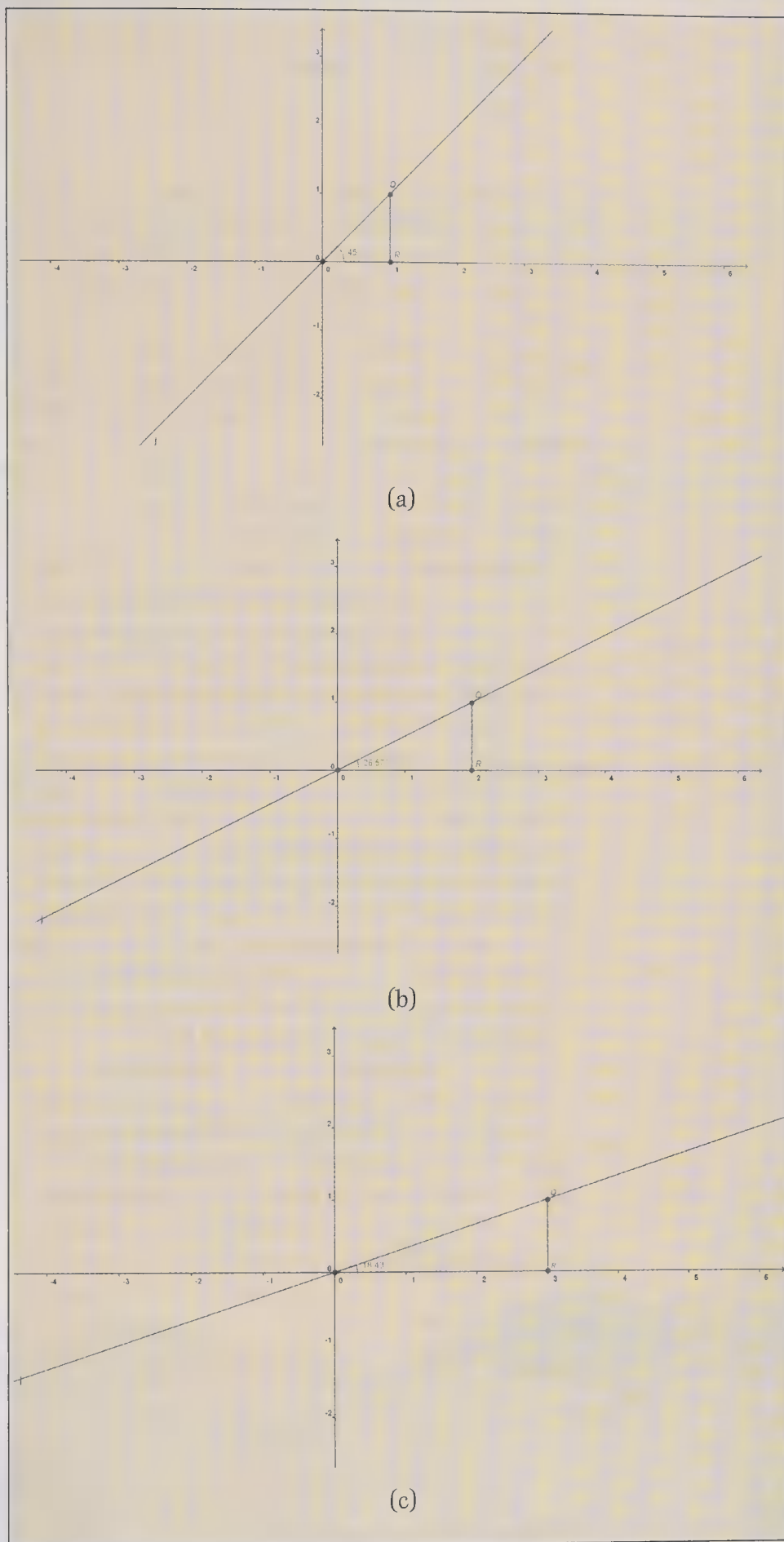


Fig. 4 As the run increases but the rise remains constant, the line becomes less steep and the angle of inclination decreases.

ratio confirms that the slope of each successive line decreases. At the same time, the angle of elevation between each line and the x -axis also decreases (see **fig. 4**). This relationship can be made explicit by asking questions such as these:

- How does the steepness of the line affect slope?
- How does the steepness of the line affect the angle that it forms with the horizontal?

After completing part 1, students should be able to explain that a line's rise-to-run ratio is associated with its steepness as well as its angle of elevation. Hence, when comparing graphs on homogeneous coordinate axes (those that have equal scaling and units on both horizontal and vertical axes), students need to understand that smaller values for slope (i.e., smaller rise-to-run ratios) are associated with flatter lines and decreased values for the angles of elevation.

Part 2: Angles of Elevation and Depression for a Hot-Air Balloon Scenario

The next set of exercises applies the ideas used in part 1 to a physical situation involving an observer watching a rising hot-air balloon. These exercises illustrate the importance of relating angles to steepness in physical situations where the "line" formed by two points (objects) is not a physical line but the line of sight between the observer and the balloon. Students apply their knowledge to realize that, as the balloon rises, the line PQ (see question 5 from the appendix, p. 278) becomes steeper and the angle of elevation becomes larger. Students can be asked to determine the largest possible value for the angle of elevation in this situation. If the balloon continues to rise straight up, line PQ will continue to get steeper, and the angle of elevation will get closer to 90° . Even so, the angle will never reach 90° ; the line may approach but will never reach a vertical line.

In problem 5(d), students are asked to consider the mathematical relationship between alternate interior angles to make sense of the physical situation. In particular, the idea of the angle of depression is introduced by considering the hot-air balloon pilot's line of sight down to the observer. The

horizontal lines are cut by a transversal—the line of sight between the observer and the pilot.

Depending on the ordering of the curriculum, students' prior mathematical knowledge may help them recall that the alternate interior angles, θ_1 and θ_2 , must be congruent. If this concept has not yet been addressed, problem 5(d) lays the foundation for students in subsequent classes to make connections between topics addressed in algebra and topics traditionally addressed in geometry. Either way, students should be able to make sense of the situation and ultimately reason that the observer's angle of elevation and the pilot's angle of depression must have the same measure.

Part 3: Integrating Mathematical and Physical Interpretations

After students have been introduced to the mathematical and physical interpretations of an angle of elevation and its relation to the slope of the line connecting two points or objects, part 3 provides additional opportunities for students to make connections between interpretations and between the physical and mathematical problem situations. These exercises are particularly important for helping students begin to relate these ideas and view them as alternative ways to describe the same notion.

Question 6 allows students to link the "rise over run" interpretation of slope with the notion that slope is related to angle of elevation. They do so by converting the maximum acceptable rise-over-run ratio to a maximum possible angle of elevation.

The idea of angle of depression from part 2 is extended to the case of ramps in question 7. This problem has less mathematical scaffolding; here, then, students must recognize the same mathematical scenario of two parallel lines cut by a transversal. Because wheelchairs must be able to travel up and down a ramp, the angle of depression is a natural extension of this problem situation.

Question 8 requires students to recognize equivalent rise-over-run ratios and apply what they have learned to realize that equivalent slopes will yield equal angles of elevation in this physical context. Question 9 addresses the misconception that lines' slopes and angles of elevation are in proportion.

As a follow-up, students could be encouraged to consider an angle of elevation of 45° versus one approaching 90° . Under a homogeneous coordinate system, a 45° angle of elevation would be associated with the line $y = x$. Doubling the slope of this line yields $y = 2x$ but does not double the angle of elevation. Even multiplying the slope by any large positive factor will never cause the angle of elevation to reach 90° .



EXTENSION: THE TANGENT FUNCTION

Although the activities presented here are intended for beginning algebra students, the effect of introducing students to alternative notions of slope can extend into their later studies. In particular, the trigonometric concept of slope can be more fully developed once students have learned the trigonometric ratios. For example, students can link the formula for slope— $m = \text{rise/run}$ —with the formula for the tangent of an angle— $\tan \theta = \text{opposite leg/adjacent leg}$ —to relate more formally the angle of elevation to the rise-over-run ratio.

For instance, plotting the relationship between a line's angle of elevation (measured in degrees along the horizontal axis) and the line's slope (measured along the vertical axis) yields the scatter plot in **figure 5**, which students may recognize as the graph of the tangent function. Moreover, students can relate the undefined behavior of $\tan \theta$ at $\pm 90^\circ$ with the undefined slope of vertical lines.

Thus, by linking the algebraic and trigonometric concepts, students have opportunities to strengthen their ideas of slope. Although introductory algebra instruction can set the stage by first drawing connections between the angle of elevation and a line's steepness, these connections can continue to strengthen students' understanding of slope throughout their educational experiences.

UNIFYING THE CONCEPT

The activities presented in this lesson explicitly encourage students to make connections between the informal concept of steepness and the mathematical concept of slope, allowing introductory algebra students to make sense of interpretations of slope as the tangent of the angle of elevation of a line. Because previous research suggests that students naturally measure steepness in terms of angles, linking slope to angles provides a direct pathway by which to connect the notions of steepness and slope. In addition to broadening students' interpretations of slope, these activities also give students a foundation for reasoning in trigonometry.

By leading algebra students to think of slope beyond the rise-over-run ratio, or $(y_2 - y_1)/(x_2 - x_1)$, teachers can help them develop a broader and deeper understanding of slope, foster skills and reasoning needed for more advanced mathematics, and build connections between mathematical ideas consistent with NCTM's goals for sense making.

REFERENCES

Moore-Russo, Deborah, AnnaMarie Conner, and Kristina I. Rugg. 2011. "Can Slope Be Negative in 3-Space? Studying Concept Image of Slope through Collective Definition Construction." *Educational*

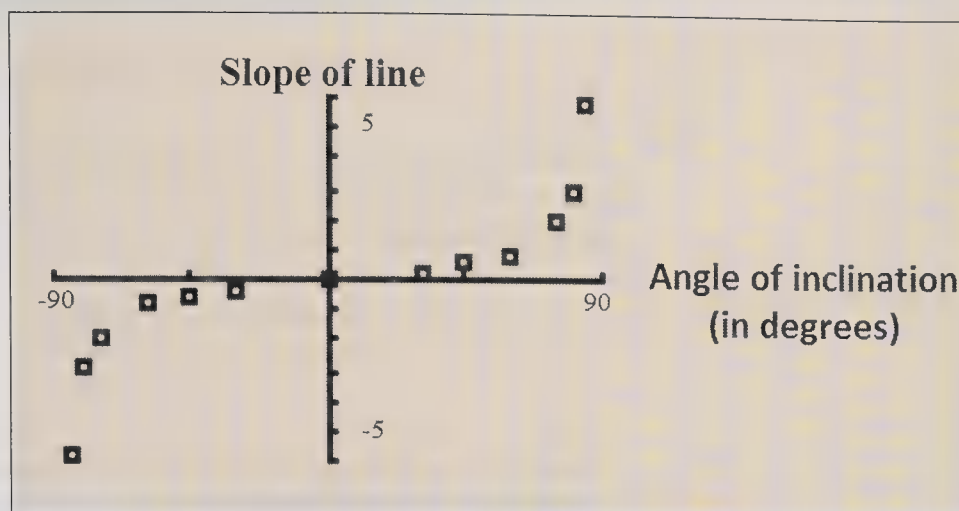


Fig. 5 A scatter plot of the relationship between the angle of inclination measured in degrees and the slope of the line resembles the graph of the relationship $y = \tan x$.

Studies in Mathematics 76 (1): 3–21.

National Council of Teachers of Mathematics

(NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

—. 2006. *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*. Reston, VA: NCTM.

—. 2009. *Focus in High School Mathematics: Reasoning and Sense Making*. Reston, VA: NCTM.

Pirie, Susan, and Lyndon Martin. 2000. "The Role of Collecting in the Growth of Mathematical Understanding." *Mathematics Education Research Journal* 12 (2): 127–46.

Stanton, Michael, and Deborah Moore-Russo. 2012.

"Conceptualizations of Slope: A Look at State Standards." *School Science and Mathematics* 112 (4): 241–48.

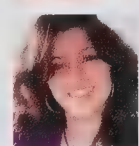
Stump, Sheryl. 1999. "Secondary Mathematics Teachers' Knowledge of Slope." *Mathematics Education Research Journal* 11 (2): 124–44.

—. 2001. "High School Precalculus Students' Understanding of Slope as Measure." *School Science and Mathematics* 101 (2): 81–89.

Tall, David, and Shlomo Vinner. 1981. "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity." *Educational Studies in Mathematics* 12 (2): 151–69.



COURTNEY R. NAGLE, crt12@psu.edu, is an assistant professor of mathematics education at Penn State Erie, The Behrend College. She is interested in the development of mathematics concepts, including the notions of slope and limit.



DEBORAH MOORE-RUSSO, dam29@buffalo.edu, teaches in the Graduate School of Education, State University of New York (SUNY), University at Buffalo. Her primary research interests include spatial visualization, communication, and reasoning.

APPENDIX: SLOPE

Part 1

- Construct point P at $(0, 0)$ and point Q at $(1, 1)$. Construct a line l passing through both points. Place point R at $(1, 0)$ and construct segments between each of the three pairs of points to form a right triangle whose hypotenuse falls along line l and whose horizontal base lies on the x -axis.
 - Find the length of a height of triangle PQR . What is the length of the corresponding base? How do these values relate to the slope of line l ?
 - Find the angle, θ , formed by the x -axis and line l .
- Next, drag the point Q horizontally to $(2, 1)$. Move R to $(2, 0)$ to maintain a right triangle.
 - Find the length of a height of $\triangle PQR$. What is the length of the corresponding base? Use these values to determine the slope of line l .
 - What is the measure of θ for this triangle?
- Next, drag the point Q horizontally to $(3, 1)$. Move R to $(3, 0)$ to maintain a right triangle.
 - Find the length of a height and a corresponding base of $\triangle PQR$. Determine the slope of line l .
 - What is the measure of angle θ for this triangle?
- Continue dragging the point Q horizontally to the right and keep moving R to maintain a right triangle.
 - As Q moves to the right, what happens to the height of the resulting right triangle? What happens to the corresponding base?
 - Use part 4(a) to explain what happens to the slope of line l as Q moves to the right.
 - What happens to angle θ as Q moves to the right? Explain.

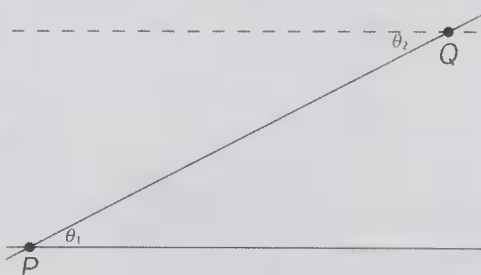
Part 2

- Suppose that a person standing at point P is observing a hot-air balloon located at point Q . The angle formed by the horizontal and the line of sight to the balloon is called the *angle of elevation*. Suppose that the balloon starts at a point Q and rises straight upward.



- As the hot-air balloon rises, what happens to the slope of the line passing through points P and Q ?
- As the hot-air balloon rises, what happens to the angle of elevation between the observer and the balloon?
- What is the largest possible value for the angle of elevation? Explain how you know.

The pilot of the hot-air balloon looks down at the observer standing at point P . The angle formed by the horizontal and the line of sight of the pilot looking down to point P is called the *angle of depression*.



- At any instant, how does the observer's angle of elevation compare with the pilot's angle of depression? Explain how you know.

Part 3

- The maximum acceptable incline for a wheelchair ramp intended for public access is 1:12. This ratio represents a vertical change of 1 unit for every horizontal change of 12 units.

- (a) Using the recommendation above, determine the maximum acceptable slope of a wheelchair ramp.
- (b) Use GeoGebra to sketch a line with the slope that you found in 6(a). What is the angle of elevation of this line?
- (c) Why might builders describe ramps in terms of their angle of elevation rather than their “rise” or “run”?
- (d) Using the recommendations, describe the acceptable angles of elevation for a wheelchair ramp.

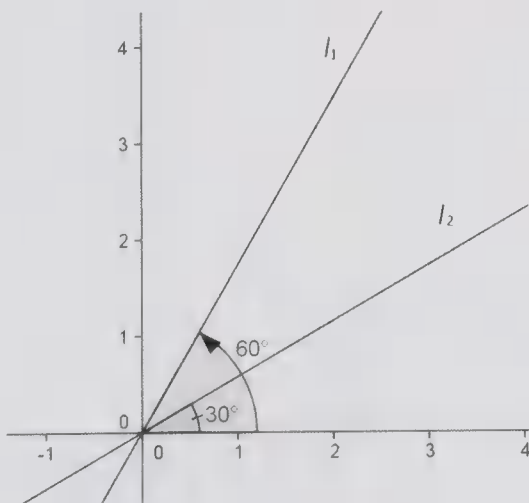
For a wheelchair ramp for the elderly, the recommended maximum incline is 1 : 18.

- (e) Which ramp is steeper—a ramp with a 1 : 12 ratio or a ramp with a 1 : 18 ratio?
- (f) Suppose that a ramp must be constructed to reach a doorway that is 3 feet above the ground. Using the 1 : 12 recommendations, determine how far from the base of the entryway the ramp must be built. Using the 1 : 18 recommendations, determine how far from the base of the entryway the ramp must be built.

7. Consider the ramp illustrated below.



- (a) Does this ramp meet the recommendations?
 - (b) What is the ramp’s angle of elevation?
 - (c) Suppose that a person in a wheelchair is at the top of the ramp, about to wheel down the incline. What is the angle of depression of the ramp?
8. Jared builds a skate ramp that is 9 feet long and 3 feet high. Kasey builds a ramp that is 15 feet long and 5 feet high.
- (a) Compare the incline of the ramps.
 - (b) Determine the angle of elevation of both ramps. What do you notice?
9. Brian claims that line l_1 is twice as steep as line l_2 (see the fig. below) because the angle of inclination is twice as large. Use GeoGebra to explore this situation. Is Brian right?



more4U

This activity sheet is available to teachers as a Word document that can be copied and edited for classroom use; go to www.nctm.org/mt.

Beginning Algebra:

Teaching Key Concepts

In keeping with the Focus Issue theme, each of this month's problems can be solved using algebra.

Determine the number of dimples on a particular golf ball given that the hundreds digit and the tens digit are the same prime number, the sum of all three digits is 12, and the number is divisible by 7.

The oldest child in a family is 5 years older than her brother. The mother is 1 year younger than twice the sum of her children's ages, and the father's age equals the mother's age plus the son's age. If the sum of all four ages is three times a prime number and the mother was younger than 50 when she had her second child, how old is the daughter?

Today's date,

11/12/13,

can be the continued ratio of the angles of a triangle that results in all three angles being integers. Determine the next date consisting of three consecutive integers that can also be used as the continued

Using the fact that

$$x^a \cdot x^b \cdot x^c = x^{a+b+c},$$

create a 3×3 magic square in which the products of the rows, columns, and diagonals are the same number. The numbers must be unique and positive but not necessarily integers.

We know that 4 tablespoons equal $\frac{1}{4}$ cup. How many tablespoons will equal $\frac{2}{3}$ cup?

A game using nickels, dimes, and quarters requires that there must be twice as many dimes as nickels and twice as many quarters as dimes. How many coins would you need to create the largest possible amount less than \$12.00?

Arrange 127 pennies into 7 stacks so that combining the completed stacks of coins can form every possible amount from 1¢ to 127¢. How many coins will be in the largest stack?

In football, points are awarded as follows:

6 points for a touchdown, 1 point for a kicked conversion, 2 points for a run or passed conversion, 2 for a safety, and 3 for a field goal. What are the only natural numbers less than 45 that cannot be a team's total score in a football game?

The mixed number $2\frac{1}{2}$ means $2 + \frac{1}{2}$. The mixed decimal

$$0.2\frac{1}{2} = 2\frac{1}{2} \cdot \frac{1}{10} = 0.2 + \frac{1}{20} = \frac{4}{20} + \frac{1}{20} = \frac{5}{20} = \frac{1}{4}.$$

Find the common fraction for the mixed decimal $0.45\frac{5}{11}$.

Compute the following product:

$$(x + a)(x - b)(x + c)(x - d) \cdots (x - z)$$

How many ways can you split a 6-in. ruler if divisions can be made only at the inch marks? You may split the ruler into as many pieces as you wish. (The split at the 1-in. mark is considered different from the split at the 5-in. mark.)

November 3 is the last of seven days of this year for which the product (month $\cdot$ day) divides the year (2013). Find the other six such dates for which such divisibility occurs in this year as well as the next year in which this divisibility occurs on exactly seven dates in the year.

Jim had an appointment at 1:00 p.m. at a location 50 miles from his home. Driving at the speed limit of 30 mph, he arrived 20 minutes late. What time did Jim leave his house?

A group of friends decides to meet for pizza, and one person calls ahead to order the pies. However, three of the group do not show up at the pizza parlor. The friends who do show up figure that if each adds \$3 to their original share, they could cover the bill, which totals \$70. How many friends eat pizza?

The sides of a triangle all have integral measure and can be represented by the expressions

$$x + 10, 2x - 3, \text{ and } 4x.$$

For how many values of x will the sum of the two longer sides be divisible by the

Solve for all positive integers (x, y) that satisfy

$$x^2 - 4xy + 3y^2 = 17.$$

16

The sum of a positive real number and six times its reciprocal is equal to its cube. What is the square of the number?

20

The following problem also comes from *Everyday Algebra*: “At a county fair John tried a game of target shooting. He received 10¢ for each hit but had to pay 5¢ each time he missed. If 30 shots cost him 15¢, how many hits did he score?”

24

Simplify the following expression:

$$\frac{x^2 - 3x - 18}{x^2 - 8x + 12} \cdot \frac{x^2 - 4x + 3}{x^2 + x - 2} \div \frac{x^2 - 9}{x^2 - 4}$$

28

A line is given by the equation $y = 2x + 6$. A second line has the same y -intercept but is perpendicular to the first line. Find the distance between the x -intercepts of these two lines.

17

Jeremiah Day, president of Yale College from 1817 to 1846, wrote *Introduction to Algebra*, which went through at least seventy editions. The following problem comes from his textbook: “If a certain number is divided by 12, the quotient, dividend, and divisor, added together, will amount to 64. What is the number?”

21

Squares of two different sizes are placed in two piles. Pile A has 7 large squares and 3 small ones. Pile B has 5 squares of each size. The total area of the squares in pile A exceeds that of pile B by 50 in.². If the sides of both squares are integers, how much longer is a side of the larger square than a side of the smaller square?

25

The base of a parallelogram is increased by 20%, and the corresponding height of the parallelogram is decreased by 20%. By what percentage is the area of the parallelogram increased or decreased?

29

If

$$f(x) = ax^2 + bx + c,$$

determine

$$\frac{f(x) - f(-x)}{2}.$$

18

Another problem from Day’s *Introduction to Algebra* is this: “An estate is divided among four children, in such a manner that the first has 200 dollars more than 1/4 of the whole; the second has 340 dollars more than 1/5 of the whole; the third has 300 dollars more than 1/6 of the whole; the fourth has 400 dollars more than 1/8 of the whole. What is the value of the estate?”

22

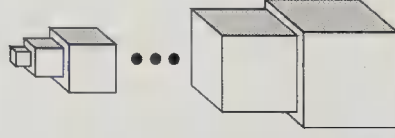
Solve the system of equations for x and y :

$$x\sqrt{2} + y\sqrt{3} = \sqrt{2}$$

$$x\sqrt{3} - y\sqrt{2} = \sqrt{12}$$

26

A 1-in. cube is centered atop a 2-in. cube that is centered atop a 3-in. cube. This pattern is continued until a 9-in. cube is centered atop a 10-in. cube. Find the height, surface area, and volume of the resulting stack of cubes.



30

Two years ago, Dana was one-fourth as old as her uncle. Two years from now, she will be one-third as old as her uncle. How old was Dana’s uncle when she was born?

19

William Betz, NCTM’s seventh president (1932–34), wrote an elementary algebra textbook entitled *Everyday Algebra*. He included the following problem: “A cork calls ‘an old-fashioned problem’: ‘A cork and a bottle together cost \$1.10. The bottle cost \$1 more than the cork. What was the cost of each?’”

23

M is the midpoint of $\overline{LN}$; $LM = x^2 - 7$; and $NM = 4x + 14$. Calculate the length LN .

27

Looking for more Calendar problems?

Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher*.

Edited by **Margaret Coffey**, Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and **Art Kalish**, artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791

Problems 17-20 were submitted by Charles Kicey. He and co-author John Seppala wrote the problems and solutions for the high school contests hosted by Valdosta (Georgia) State University, where they currently teach.

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, \$22.95) and the 100 Problem Poster (stock number 13207, \$9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.—Eds.

1. An infinite number of solutions is possible. To apply the concept of multiplication of numbers with like bases but different exponents, create a standard magic square in which the sums of the columns, rows, and diagonals are all the same. One such square is shown below.

8	1	6
3	5	7
4	9	2

Then select any nonzero positive number as the base and replace the contents of each cell with the base raised to the power of the number displayed in each cell. Doing so will create the desired result. If the base number is 2, the resulting magic square is shown below.

256	2	64
8	32	128
16	512	4

2. 1. The only score that is not possible is 1. Since 2 is the result of a safety, all even scores are possible. Since 3 is the result of a field goal, we can add multiples of 2 to achieve all remaining odd number scores.

3. Send student solutions to the Problem of the Month editors, Séan Madden (smadden1@greeleyschools.org) or Ricardo Diaz (Ricardo.Diaz@unco.edu).

4. 336. Consider first the possible primes in the hundreds digit and the tens digit. If they are both 2, the number is 228. The test for divisibility by 7 is to excise the units digit and subtract twice that digit

from the rest of the number: $22 - 2(8) = 6$. Since 6 is not a multiple of 7, neither is 228. Try 336; $33 - 2(6) = 21$, and since 21 is a multiple of 7, so is 336. The only other remaining possible answer would be 552, which is not divisible by 7.

5. $32/3$. One approach is to find the number of tablespoons in 1 cup. Since $1/4$ cup is 4 tablespoons, 1 cup is 16 tablespoons, and $2/3$ of a cup is $2/3$ of 16, or $32/3$.

Alternate solution: Set up a proportion letting x be the number of tablespoons in $2/3$ cup. Therefore,

$$\frac{x}{\frac{2}{3}} = \frac{4}{\frac{1}{4}} \rightarrow \frac{1}{4}x = \frac{8}{3} \rightarrow x = \frac{32}{3}.$$

6. $5/11$. Rewrite as follows:

$$\begin{aligned} 0.45 \frac{5}{11} &= 45 \frac{5}{11} \cdot \frac{1}{100} = \frac{45}{100} + \frac{5}{1100} \\ &= \frac{495+5}{1100} = \frac{500}{1100} = \frac{5}{11} \end{aligned}$$

Alternate solution:

$$\begin{aligned} 0.45 \frac{5}{11} &= 45 \frac{5}{11} \text{ hundredths} \\ &= \frac{500}{11} \text{ hundredths} = \frac{5}{11} \end{aligned}$$

This approach requires understanding that hundreds $\cdot$ hundredths equals 1, a concept emphasized in the Common Core State Standards.

7. 11:40 a.m. The time it takes to drive 50 miles at 30 mph = 50 miles/30 mph = $1 \frac{2}{3}$ hr. = 1 hr. 40 min. Since Jim arrived 20 min. late, he arrived at 1:20 p.m. Subtract 1 hr. 40 min. from that time to get 11:40 a.m.

8. 13. Let S = the boy's age. Then $S + 5$ = the daughter's age; $2(2S + 5) - 1 = 4S + 9$ = the mother's age; and $5S + 9$ = the father's age. Since the mother's age at the time that she had her son was less than 50, we know that the difference of their ages is less than 50. Therefore, $(4S + 9) - S < 50$, so $3S < 41$ and $S < 13 \frac{2}{3}$. The sum of all four ages is $11S + 23$, which must equal three times a prime number. If S is odd, the sum will be even and therefore not equal to three times a prime. The only possible values for S are 2, 4, 6, 8, 10, and 12. If we test each of these values, the only number that results in three times a prime number is 8. Thus, the son is 8 years old, and his sister is $8 + 5 = 13$ years old. (See solution 8 table.)

9. 63. Let n = the number of nickels, d = the number of dimes, and q = the number of quarters. Since $d = 2n$ and $q = 2d = 4n$, we can set up and solve the following equation: $5n + 10d + 25q < 1200 \rightarrow 5n + 20n + 100n < 1200 \rightarrow 125n < 1200 \rightarrow n < 9.6$. Since n must be an integer, $n = 9$, $d = 18$, and $q = 36$, and the sum is 63.

10. 0. One term in the pattern is $(x - x) = 0$, and 0 times any number is 0.

11. 7. Let f = the number of friends expected to eat pizza. Since the bill was \$70, they would each expect to pay \$70/ f . The actual number of friends who had to share the bill was $f - 3$, and each had to pay \$3 more than expected. Solve the following equation:

$$\begin{aligned} (f-3)\left(\frac{70}{f}+3\right) &= 70 \rightarrow \\ 70+3f-\frac{210}{f}-9 &= 70 \rightarrow \\ 3f^2-9f-210 &= 0 \rightarrow \\ f^2-3f-70 &= 0 \rightarrow \\ (f-10)(f+7) &= 0 \rightarrow \\ f &= 10 \text{ or } f-3=7 \end{aligned}$$

Alternate solution: The only possible values for f and $f - 3$ must be divisors of 70 (1, 2, 5, 7, 10, 14, 35, and 70). The only two numbers in the set that differ by 3 are 7 and 10. At times algebra makes the solution easier, and at other times it can make the solution more complex.

Solution 8						
S	2	4	6	8	10	12
$11S + 23$	45	67	89	111	133	155
$(11S + 23)/3$	15	22.3	29.7	37	44.3	51.7

12. 12/11/13. If we do not allow the digit 0 to be used, then 11, 12, and 13 are the only numbers that can be used in the continued ratio of integer measures for the angles of a triangle. If m , d , and y represent, respectively, the month, day, and year of the date, then $m + d + y$ must be a factor of 180, $y \geq 13$, $1 \leq m \leq 12$, and $1 \leq d \leq 31$. For the next date, y must be at least 13, meaning that m must be 11, 12, or a number larger than 12. The last choice is impossible, so we conclude that the only set of numbers that satisfies the condition of the problem is 11, 12, and 13. Rearranging the three numbers produces 12/11/13. The next time this phenomenon will occur will be in 100 years, 11/12/2113. If we allow zeros to be used, the next date is 01/02/03, which is January 2, 2103.

13. 64. The stacks should contain 1, 2, 4, 8, 16, 32, and 64 pennies. This problem is based on the binary (base 2) counting system. Every integer can be written by using 0 and 1 if the place values in the system are powers of 2. For example, the number 17 is 10001₂, meaning one 16 and one 1 but no 8s, 4s, or 2s.

14. 31. Consider the 5 marks between the two ends of the ruler. Each mark can be used to cut the ruler or can be left uncut. Therefore, we have 2 possibilities at each mark and a total of $2^5 = 32$ possibilities. If the ruler is not cut at any of the points, then we would not consider the ruler separated into two or more parts, so we must subtract 1 from 32 to get 31.

Alternate solution: Consider the number of ways to cut the ruler into two parts. We can do so in 5 ways since we can make the cut at exactly one of the 5 marks. Then consider the number of ways to cut the ruler into three parts. The 10 possible ways are $\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$. There are also 10 ways to cut the ruler into four parts; they can be seen as not cutting at the points in the previous set. There are

5 ways to cut the ruler in five parts by leaving exactly one of the 5 marks uncut. There is only 1 way to cut it into six parts. The sum of $5 + 10 + 10 + 5 + 1$ is 31. You might notice a connection between this set of results and Pascal's triangle.

15. 2. Begin by applying the notion that the sum of the two shortest sides of a triangle must be greater than the third side. We do not know which is the longest side, so we must solve each of these inequalities: $3x + 7 > 4x$; $5x + 10 > 2x - 3$; and $6x - 3 > x + 10$. The corresponding results are $x < 7$; $x > -13/3$; and $x > 13/5$. The only integers satisfying these inequalities are 3, 4, 5, and 6. Substitute these values for x to get the possible lengths of the sides of the triangles: $\{(13, 3, 12), (14, 5, 16), (15, 7, 20), (16, 9, 24)\}$. The only possibilities that result in the sum of the longer sides being a multiple of the shortest side occur when $x = 4$ and $x = 5$. Thus, there are only two solutions.

16. (25, 8). Factor the given equation: $x^2 - 4xy + 3y^2 = (x - 3y)(x - y) = 17$. Since 17 is prime, one factor must be 17, and the other must be 1. Solve the two resulting systems of equations:

$$x - y = 17 \text{ and } x - 3y = 1$$

or

$$x - y = 1 \text{ and } x - 3y = 17.$$

In each case, subtracting the two equations gives $2y = \pm 16$ or $y = \pm 8$. The problem asks for the positive values; when $y = 8$, $x = 25$. (Note that the graph of the given equation is a hyperbola with $y = x$ and $3y = x$ as asymptotes.)

17. 15. Find the x intercept of the first line: $y = 2x + 6 \rightarrow 0 = 2x + 6 \rightarrow x = -3$. The equation of the second line must be $y = -x/2 + 6$. Find the x -intercept: $0 = -x/2 + 6 \rightarrow x = 12$. The distance from -3 to 12 is 15.

Alternate solution: Consider the more general problem, in which the first line is given by $y = mx + b$, where $m, b > 0$. The equation of the second line is $y = (-1/m)x + b$. The x -intercepts are $(-b/m, 0)$ and $(bm, 0)$, respectively. The distance between the intercepts is $bm - (-b/m) = (b/m)(m^2 + 1)$, which is always positive. For $m = 2$ and $b = 6$, the distance is 15.

18. bx . Substitution gives

$$\frac{f(x) - f(-x)}{2} = \frac{(ax^2 + bx + c) - (ax^2 - bx + c)}{2} = bx.$$

19. 24. Let d and u be Dana's and her uncle's present age, respectively. We have $d - 2 = (u - 2)/4$ and $d + 2 = (u + 2)/3$. Subtracting the two equations to solve the system, we obtain $4 = u/12 + 7/6 \rightarrow u = 34, d = 10$. Since the difference of their ages is constant, $u - 10 = 24$.

20. 3. If x is the unknown number, it must satisfy $x + 6/x = x^3 \rightarrow x^4 - x^2 - 6 = 0$. Factoring gives $(x^2 - 3)(x^2 + 2) = 0$, so $x^2 = 3$. The number is real, so we discard the imaginary solution.

21. 48. Let x be the unknown number. Then $x/12 + x + 12 = 64$. Combine like terms to obtain $13x/12 = 52$, implying that $x = 52(12/13) = 48$.

22. \$4800. Let v = the unknown value of the estate. Then

$$200 + \frac{1}{4}v + 340 + \frac{1}{5}v + 300 + \frac{1}{6}v + 400 + \frac{1}{8}v = v.$$

We have

$$v \left(\frac{30}{120} + \frac{24}{120} + \frac{20}{120} + \frac{15}{120} \right) + 1240 = v \rightarrow \frac{31}{120}v = 1240 \rightarrow v = 1240 \left(\frac{120}{31} \right) = \$4800.$$

23. \$0.05 for the cork, \$1.05 for the bottle. Let c = the cost of the cork. Then the bottle costs $c + 1$. We have $c + c + 1 = 1.1 \rightarrow 2c = 0.1 \rightarrow c = \0.05 . So the cork costs 5 cents, and the bottle costs \$1.05.

24. 9. Let h = the number of hits. Then $30 - h$ = the number of misses. John gains $10h$ and loses $5(30 - h)$; the difference is a loss of 15¢. We have $10h - 5(30 - h) = -15 \rightarrow 15h - 150 = -15 \rightarrow 15h = 135 \rightarrow h = 9$.

25. 1 in. Let the area of the large square be L , and let the area of the small square be S . Then $(7L + 3S) - (5L + 5S) = 50 \rightarrow 2L - 2S = 50 \rightarrow L - S = 25$. Let a side of the larger square be x and a side of the smaller square be y . Then $x^2 - y^2 = 25 \rightarrow (x - y)(x + y) = 25$. Since the sides must be integers, $x - y$ must be a factor of 25. The only factors of 25 are 1, 5, and 25. We see that $x - y < x + y$, so $x - y = 1$.

26. $x = 8/5$ and $y = -\sqrt{6}/5$. Multiply the first equation by $\sqrt{3}$ and the second equation by $\sqrt{2}$ to produce

$$x\sqrt{6} + 3y = \sqrt{6} \text{ and } x\sqrt{6} - 2y = \sqrt{24} = 2\sqrt{6}.$$

Subtract the second equation from the first:

$$5y = -\sqrt{6} \rightarrow y = -\sqrt{6}/5$$

Substitute this into one of the original equations to find $x = 8/5$.

27. 4 or 84. Since M is a midpoint, the two segments LN and NM are equal. Thus, $x^2 - 7 = 4x + 14 \rightarrow x^2 - 4x - 21 = 0 \rightarrow (x + 3)(x - 7) = 0 \rightarrow x = -3$ or $x = 7$. Substitute -3 to get $MN = 4(-3) + 14 = 2$, and substitute 7 to get $MN = 4(7) + 14 = 42$. These are each half of LN , so LN is either 4 or 84.

28. 1. We factor as follows:

$$\frac{x^2 - 3x - 18}{x^2 - 8x + 12} \cdot \frac{x^2 - 4x + 3}{x^2 + x - 2} \cdot \frac{x^2 - 4}{x^2 - 9} = \frac{(x - 6)(x + 3)}{(x - 6)(x - 2)} \cdot \frac{(x - 1)(x - 3)}{(x - 1)(x + 2)} \cdot \frac{(x - 2)(x + 2)}{(x - 3)(x + 3)}$$

Since each factor in the numerator also appears in the denominator, we can reduce the fraction to 1.

29. 4% decrease. When the base is increased by 20%, it is 1.2 times the original base. When the height is decreased by 20%, it is 0.8 times the original height. The area of a parallelogram is the product of the base and height, so the new area is $(1.2)(.8) = 0.96$ times the original area. Thus, the area has been reduced by $0.04 = 4\%$.

30. Height = 55 in., area = 1740 in.², and volume = 3025 in.³. The height is the sum of the integers $1 + 2 + 3 + \dots + 10 = 10(10 + 1)/2 = 55$. The area is 4 times the sum of the area of a face of each cube + the area of the top and the bottom of the stack. The sum of the areas of the 10 faces = $1^2 + 2^2 + 3^2 + \dots + 10^2 = 10(11)(2 \cdot 10 + 1)/6 = 385$. The areas of all the tops combine to equal the area of the bottom = 100. Thus, the total surface area = $4(385) + 2(100) = 1740$. The volume of the stack is the sum of the cubes = $1^3 + 2^3 + 3^3 + \dots + 10^3 = (1 + 2 + 3 + \dots + 10)^2 = 55^2 = 3025$. In general,

$$\sum_1^n i = \frac{n(n+1)}{2},$$

$$\sum_1^n i^2 = \frac{n(n+1)(2n+1)}{6},$$

and

$$\sum_1^n i^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Understanding and Teaching Statistics and Probability

One of every two children today lives in poverty.

Six billion hours of videos are watched every month on YouTube®.

Data are everywhere—being generated, collected, and mined for all kinds of purposes. To make and interpret decisions in today's world, we all must be statistically literate. Recent curricular policy documents (e.g., CCSSM, GAISE, MET 2) increased the importance of statistics and probability as pillars of the secondary school mathematics curriculum. How does this increased attention to data and chance affect your teaching and your students' learning?

The Editorial Panel of *Mathematics Teacher* invites teachers to share their experiences teaching statistics and probability. We particularly encourage submissions that help other *MT* readers gain new perspectives on dynamic approaches involving students in the process of wrestling with data and chance.

- How do you harness students' interests to engage in authentic statistical inquiry?
- What kinds of phenomena have you or your students found interesting, provocative, or especially productive in supporting statistical reasoning?

- How do you motivate students to generate their own statistical questions?
- What have you learned about the ways in which students design simulations that model statistical or probabilistic phenomena?
- What problems and solutions have you discovered when students are collecting, analyzing, and interpreting data from statistical studies?
- In what ways have you channeled the power of technology to enhance your students' understanding of probability and statistics?

You may submit your completed manuscript for review by accessing **mt.msubmit.net**. Indicate that the manuscript is being submitted in response to the call Statistics and Probability. Be sure to enter the call's title in the Department/Calls field. No author identification should appear in the text of the manuscript. For additional guidelines for preparation of manuscripts, see www.nctm.org/publications/content.aspx?id=22602.

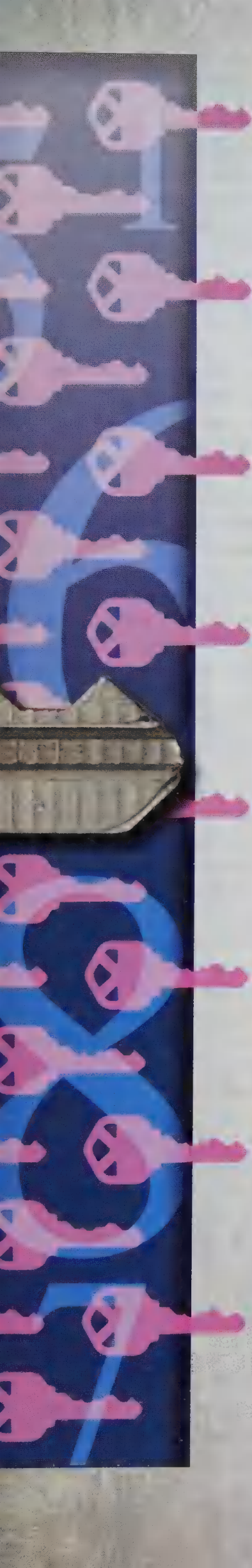


NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

MATHEMATICS
teacher

CALL FOR MANUSCRIPTS





Problems before Procedures: Systems of Equations

Students today come to first-year algebra with considerable prior experience and a wide range of skills. Teachers need to modify their instructional strategies accordingly.

Kasi C. Allen

As a new teacher in the 1980s, I savored my work with ninth graders who were studying beginning algebra. I especially enjoyed the chance to excite young minds about the wonder of variables and the power of generalization. We programmed graphing calculators (a tool most of them were encountering for the first time) and explored the role of mathematics in other disciplines, especially art and science. My ninth graders were proud to study algebra. They approached the subject with some trepidation but also with curiosity. What was this algebra thing they had heard so much about?

That was then; this is now. Today, beginning algebra in the high school setting is associated more with remediation than pride. Students

enroll by mandate and attend under duress. Class rosters in this “graveyard” course, as it is often referred to, include sophomores and juniors who are attempting the course for the second or third time. Even the ninth graders have seen many of the ideas in an earlier context because of the increasing prevalence of seventh- and eighth-grade mathematics courses designed to frontload first-year algebra topics. As a result, the content can feel like review, even when it is not. Such circumstances present unique challenges for teachers and students alike.

My experience as a mathematics educator leads me to a single principle that can consistently lead to successful teaching and learning in an early algebra class: Pose problems before presenting procedures. This is not a new idea. However, it is one that many teachers either

Table 1 Record of Student Guesses

First number	20	21	22	23	24	25	26
Second number	15	14	13	12	11	10	9
Sum	35	35	35	35	35	35	35
Difference	5	7	9	11	13	15	17

forget or opt not to implement under the pressures of time and topic coverage. Especially when accompanied by instructional practices that prioritize student communication and collaboration, this subtle shift in practice can prove particularly valuable in the beginning algebra setting. Compelling problems provide a powerful tool for assessing prior knowledge, encourage students to generate mathematical ideas, and pave the path to symbolic representations. Using problems to drive mathematics instruction also levels the playing field by valuing powerful thinking over memorization, thus fostering a community of learners in which everyone has something to contribute. This article uses systems of linear equations to illustrate how this strategy might unfold in a classroom.

ASSESSING PRIOR KNOWLEDGE: THE MYSTERY NUMBERS PROBLEM

Students bring extensive prior knowledge to beginning algebra, much more than even they know that they possess. The challenge is to uncover this understanding in ways that students find safe, empowering, and relevant to the topic at hand. In the case of systems of equations, students will likely have had some recent experience with equations in two variables. They should also have a sense of the multiple ways to represent such relationships: tables, graphs, symbols, and verbal description.

Many textbooks begin immediately with symbols. Instead, we start with a problem like the Mystery Numbers problem:

I'm thinking of two numbers. Their sum is 35, and their difference is 13. What are the numbers?

As teachers, we recognize that the Mystery Numbers problem lends itself to representation with two equations and two variables. However, whether or not students have prior experience in solving systems of linear equations, they will take a nonsymbolic approach to this problem.

In class, I recommend beginning with a think-pair-share strategy: Give each student some time to consider the problem alone and to jot down initial ideas. Then organize students in pairs to share their thinking and work toward a solution. Ask pairs to keep track of the strategies they discuss, focusing

on process at this point. For pairs that seem to finish quickly, ask how many solution methods they can find.

Once all pairs of students seem to have found a solution, move to pair consultations: Have each pair meet briefly with another to share strategies and solutions. These small groups should select one reporter who can communicate the collective ideas during the whole class discussion that will follow. As groups report out, teachers can provide a public posting place or "landing pad"—such as poster paper, whiteboard, or SMART Board®, which can be displayed and added to during the conversation as well as saved for future reference.

For the Mystery Numbers problem, students will likely describe a process of comparing number pairs, such as 20 and 15, that add up to 35 and then finding the difference, making adjustments until they find the right combination. With some encouragement, either during pairs work or as part of the whole-class debriefing, students might record their thinking in a table. If they do not, then the teacher can suggest making a table to keep track of the number pairs that students have tested (see **table 1**). The table will encourage new questions: What patterns do students see in the table? What do they notice about the relationship between the first number and the second number? How does one affect the other? Does this problem have many answers or just one? How do you know?

The purpose of carefully examining the table is to ensure that students get some sense of how all the possible mystery numbers are related to one another and why only one pair can serve as the answer to the given problem. Students can use this thinking to represent the situation symbolically.

If the class discussion indicates that students have used a systematic guess-and-check strategy exclusively, then teachers can ask students how they might use variables to solve this problem. Pairs can then work together to represent each of the given facts with an equation using variables—one for each fact. In situations such as this, index cards work well as a reporting tool. On the index card, pairs write down the equations they come up with, along with their names. Teachers can collect these index cards as they move about the room, make a quick formative assessment, and decide where they want to take the group next. (Index cards can also be used at the end of the class period as an exit ticket to provide assessment data that will inform the design of the next day's lesson.)

If, on the other hand, a significant number of students immediately solve the problem using two equations and an algebraic strategy (such as elimination or substitution), ask students to explain how their strategy works. Remind them how important

it is to understand every procedure or algorithm they choose to use. Try to distinguish between what students know and what they have simply memorized.

In either case, students may benefit from exploring one or two more similar problems before making the move to symbols. Here are two possibilities:

I'm thinking of two numbers. Their sum is 111, and their difference is 55. What are the numbers?

I'm thinking of two numbers. The larger number is double the smaller one. Their sum is 96. What are the numbers?

GENERATING MATHEMATICAL IDEAS

The ultimate goal of posing problems before presenting procedures is to help students generate mathematical ideas based on their own thinking rather than telling them what to think. With the Mystery Numbers problem, students need to see for themselves that two conditions must be satisfied simultaneously: The numbers must add up to 35 and have a difference of 13. Infinitely many pairs satisfy each condition individually, but only one pair satisfies both conditions at the same time. If students have prior experience working with variables and writing equations, then (perhaps with some gentle nudging and careful questioning) they should be able to represent the two conditions with two equations:

$$\begin{aligned}x + y &= 35 \\ x - y &= 13\end{aligned}$$

Once the class has generated the equations above and agreed on their meaning, including what the variables represent, students can begin using their mathematical know-how to determine the values for x and y that make these equations true at the same time. In their pairs, students can talk about

this idea for a few minutes. What possibilities do they see? Before pairs report out, divide the class in half. Let students know that half the pairs will pursue a graphing strategy, while the other half will take a more algebraic approach. As a class, brainstorm helpful hints for each group. (The activity described here is intended for a long block of eighty to ninety minutes.)

Some ideas that teachers will want to help elicit for a graphing approach include these:

- Both equations can be graphed as lines.
- Consider rewriting the equations in a different form to help with graphing.
- What do you see when both lines are graphed on the same set of axes?

Some ideas that teachers will want to help elicit for an algebraic approach include these:

- Rewrite the equations to see whether you can connect them in some way.
- Use properties of equality. Example: If $a = b$ and $c = d$, then $a + c = b + d$.
- If you find the value of one variable, how can it help you find the other?

Students can test and record their strategies while the teacher circulates to ask clarifying questions and provide support (see **fig. 1**).

Student pairs likely will generate powerful ideas here that clearly point to the procedures traditionally taught for solving a system of two linear equations. Pairs can write up their work on an 11×17 piece of paper, creating a miniposter that shows the connection between their numeric solution to the problem and their algebraic or graphing solution as well as any questions that they have. If time permits, pairs can consult with one another and hang their miniposters on one wall of the classroom for a gallery walk, in which students walk past each poster, giving the class a greater sense of its

Graphing Approach	Algebraic Approach
• How do you know that the graphs and the equations represent the same set of numbers? • Why does it make sense that your lines intersect? • How can you anticipate an intersection by looking at the equations? • Where do your lines intersect? • What is the significance of this point?	• Can you remind me what x and y represent in these equations? • Do x and y have the same meaning in both equations? How do you know? • How might you combine the two equations to make one equation? • Explain in words what this statement means: "$\text{If } a = b \text{ and } c = d, \text{ then } a + c = b + d.$" • How does this statement apply to our equations?

Fig. 1 Teachers can ask questions such as these to elicit understanding.

collective work. A short whole-class discussion wraps up the lesson.

BUILDING TOWARD UNDERSTANDING OF PROCEDURES

As the work continues in the days ahead, a focus on solving incrementally more difficult problems will verify and enrich student conjectures. Delaying the presentation of procedures ultimately creates a situation in which articulating algorithms brings together the collection of student-initiated ideas that have surfaced in class discussion. When students feel empowered to generate the mathematics themselves, rather than simply memorize or repeat what others have created, mathematics takes on new meaning, and students become intrinsically motivated to learn (Middleton and Jansen 2011). In addition, the problem-solving process enables students to consider techniques that rarely surface in a class focused first on procedures and then on problems as applications of those procedures.

The following example is based on the Farmer's Market problem, which appeared in *Algebra and Algebraic Thinking in School Mathematics* (Ferrucci et al. 2008).

At the local Farmer's Market, you bought 7 apples and 4 peaches for \$4.80. Your friends bought 5 apples and 2 peaches for \$3.00. When you got home, your sister asked how much each fruit cost. You don't remember, but you can use mathematics to figure it out. (Adapted from Ferrucci et al. 2008, p. 196)

This problem and others like it demonstrate how elementary school students can use the Singapore model method to solve a system of equations problem without symbols (Ferrucci et al. 2008). When older students are given this problem before formal

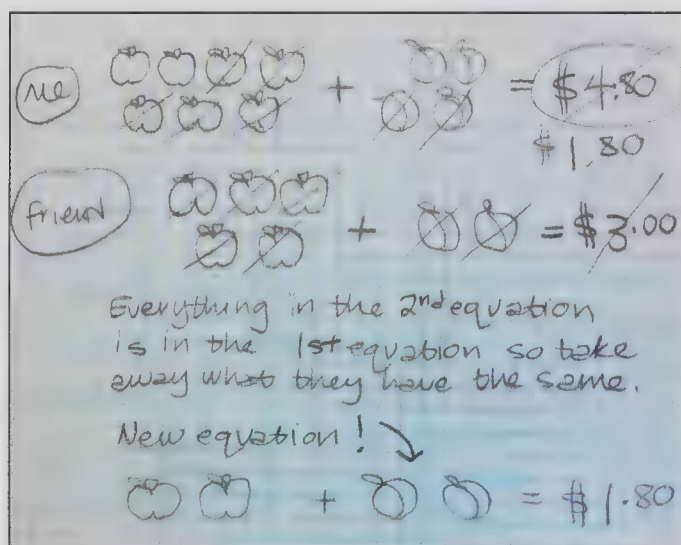


Fig. 2 A student-generated method demonstrates powerful preprocedure thinking.

instruction about solving systems of linear equations, they generate sketches and other nonsymbolic approaches akin to the model method (see **fig. 2**).

This student's ideas taught me a profound lesson about the importance of making space for mathematical creativity through problem solving. With some encouragement and probing questions, she ultimately solved the problem using her "everything in this equation is in the other equation" approach. "This is legal, right?" she asked with excitement. Her conclusion: Three apples cost \$1.20, so one apple cost 40 cents, and (after a short pause) a peach costs 50 cents. As this student reflected on her work, she also noted that her "new" equation could have been simplified:

$$\text{cost of apple} + \text{cost of peach} = 90 \text{ cents}$$

As I strive to make more room in my teaching for this kind of student thinking to occur, I marvel at the new insights I gain from my students. High-achieving students take note of their peers' methods, many of which involve highly efficient mathematical reasoning before symbolic manipulation.

DRIVING THE MOVE TO SYMBOLS

Ongoing opportunities to solve increasingly complex problems—for which strategies such as drawing diagrams, making tables, and engaging in an informed guess-and-check strategy become less practical—will stimulate student interest in more systematic and globally applicable techniques. In addition, ongoing experience as cooperative problem solvers, who readily incorporate the thinking of others into their own ideas, will enable students to view the procedures as culminating strategies, tools that encompass the thinking of many people rather than a single person—namely, their teacher.

The Spare Change Problem

This is a highly adaptable problem that resonates with students in a range of settings. It is concrete and tangible, with multiple entry points. The larger the number of coins, the more students will employ strategies using variables.

Asa saves dimes and quarters so that he will have exact change for the bus. He dumps out the container that he has been using to collecting his coins and finds that he has 47 coins that are worth \$9.50. How many of each coin does he have?

I especially like the way students see the connection between the two variables in this problem. Even those generally trying to avoid symbols create detailed tables to demonstrate how they can exchange 2 quarters for 5 dimes.

The Theater Ticket Mystery Problem

This is another problem that nudges students toward representing ideas symbolically and articulating some general principles for combining two equations in two variables.

Last night was a sellout for the school play. However, the students running the box office lost track of the number of adult tickets and children's tickets that were sold. Here's what you know: The theater seats 200 people. Adult tickets were priced at \$8, and children's tickets were \$5. At the end of the night, the box office had collected \$1360 in ticket sales. How many children and how many adults saw the show last night?

As students begin working more consistently with symbols, one common challenge with both the Spare Change and the Theater Ticket Mystery problems is that some students resist representing them with two equations. It is as if they simply do not see the problem in this way. Instead of writing a separate equation for the total number of coins or tickets, they make a mental substitution, generating a single equation in one variable. For example, they might write $25q + 10(47 - q) = 950$ for the Spare Change problem and $8A + 5(200 - A) = 1360$ for the Theater Ticket Mystery problem.

Such students do not seem to see the total number of coins or the total number of seats as facts that warrant a separate equation. It is important for teachers to understand and honor this preference. However, it is equally critical to help students recognize the value of using two equations.

In my own teaching, for each problem that students explore, we try to generate at least two, preferably three, solution methods—the more, the better. Doing so helps students see how, even in algebra, there is more than one way to arrive at the correct answer. It also creates an opportunity to discuss how the “one-equation” strategy could be viewed as a “two-equation” strategy, with some elements completed mentally rather than on paper. As the exploration of linear equations continues, well-crafted distance problems—preferably based on student experiences (such as traveling upstream versus downstream) or data collected in class (such as using stopwatches to compare student rates of walking, jumping, and running)—also help students readily embrace the two equations in two-variables strategy.

REAPING THE BENEFITS

Letting problems serve as the leading edge for mathematics instruction takes courage and careful planning, especially in a first-year algebra class. Many teachers fear that they will not be able to cover the

same amount of material; they believe that presenting procedures first ensures that all students “see the material.” In my experience, students do not learn the topics that we “cover”; they are much more likely to remember and retain the mathematics that they do and create themselves. Doing mathematics means solving problems—that is, solving compelling mathematical questions to which one does not immediately know the answer. The exercises that fill most textbooks are not *problems*. As the term *exercises* suggests, these questions rarely involve new thinking on the part of students; they merely serve as practice for procedural knowledge (Johnson and Herr 2001).

Students need something more. They need to think deeply, question decidedly, reason carefully, and verify confidently so that they can come to believe that “learning mathematics involves learning ways of thinking. It involves learning powerful mathematical ideas rather than a collection of disconnected procedures” (Carpenter et al. 2003, p. 1). Rather than viewing mathematics as something done to them, students need to view mathematics as something they can and want to do. Good problems followed by rich classroom interactions can make all the difference.

REFERENCES

- Carpenter, Thomas P., Megan L. Franke, and Linda Levi. 2003. *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School*. Portsmouth, NH: Heinemann.
- Ferrucci, Beverly, Berinderjeet Kaur, Jack A. Carter, and BanHar Yeap. 2008. “Using a Model Approach to Enhance Algebraic Thinking in the Elementary School Mathematics Classroom.” In *Algebra and Algebraic Thinking in School Mathematics*, 2008 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Carole E. Greenes and Rheta Rubenstein, pp. 195–210. Reston, VA: NCTM.
- Johnson, Ken, and Ted Herr. 2001. *Problem-Solving Strategies: Crossing the River with Dogs and Other Mathematical Adventures*. Emeryville, CA: Key Curriculum Press.
- Middleton, James A., and Amanda Jansen. 2011. *Motivation Matters and Interest Counts: Fostering Engagement in Mathematics*. Reston, VA: National Council of Teachers of Mathematics.



KASI C. ALLEN, kasi@lclark.edu, is assistant professor of mathematics education in the Graduate School of Education and Counseling at Lewis and Clark College in Portland, Oregon. She is interested in issues of mathematics equity and strategies for mathematically empowering students across the grade levels.



Tea St

Exercises about glue and trees and addresses can inoculate students against their notorious tendency to reduce incorrectly when simplifying expressions.

Ethan M. Merlin

Every teacher of experience knows that a great many of his algebra pupils all the way from the first year in high school up to college continue with almost comical regularity to make strange mistakes in the subject of ‘cancellation’ in fractions,” commented August Grossman, a teacher from St. Louis, in a previous issue of *Mathematics Teacher*. His students, he noted, make “mistakes that show clearly that the essence of the matter has escaped them. It is not necessary here to enumerate all the errors; suffice it to say that pupils are apt to cancel any two numbers or expressions that are the same, no matter in what context. For instance, all [of the following]

look equally correct to them” (Grossman):

$$\frac{3x}{3y}, \frac{3x+7}{3y+8}, \frac{3x+7}{3+7},$$

$$\frac{8(x+7)}{9(5x+7)}, \frac{3(\cancel{x+7})+11}{3(\cancel{x+7})+2}$$

Although the mistakes that Grossman describes are presumably familiar to all secondary school mathematics teachers, it is striking that Grossman made this observation in this journal’s February issue—in 1924.

Have we progressed since 1924 in teaching students simplification of rational expressions? Grossman wrote that “it is quite evident ... that

teaching structure in Algebra

the subject of ‘cancellation’ has not been broken up in a simple way into its constituent elements” and that he had seen “no such analysis in any text.” Almost ninety years later, I haven’t, either. Although I have many more curricular resources at hand than Grossman did for teaching students how to appreciate algebra’s applications, I have not come across any tools for teaching students to look at any algebraic expression and correctly perceive the “constituent elements” from which it is built.

In my own teaching, I have developed tasks for students that address the missed “essence of the matter” of algebraic transformations. Specifically, I have found that having students practice *perceiving* algebraic structure—by naming the “glue” in expressions, drawing expressions using “trees,” and describing “subexpression addresses”—all help students avoid these longstanding mistakes and, more significant, cultivate true fluency in algebra.

WHICH OPERATION IS THE GLUE?

I want students to be able to look at an algebraic expression containing multiple operations and to name the operation that glues the expression together. For example, the expressions $x + 3y$ and $2(x + 3)$ both are built of an addition and a multiplication operation. However, in $x + 3y$, the addition is the glue holding the expression together (with the multiplication subsumed in one of the terms), whereas in $2(x + 3)$ the multiplication is the glue (with the addition subsumed in one of the factors).

Some students immediately see which operation is an expression’s gluing operation, but others need help. I tell students to imagine substituting numbers for the variables and calculating until they have a single number. Next I ask, “What was the last operation you performed to find that number?”

That final operation is the expression’s glue.

Identifying the gluing operation requires knowing which operation would come last according to the sequence that we usually teach as order of operations. When I started teaching, I taught this operation hierarchy as a chronological procedure for simplifying expressions. More recently, I have realized that the so-called order of operations is far more than a recipe for ensuring that everyone gets the same answer. In fact, this hierarchy determines the very structure of expressions: It is why, for example, $2x + 3$ must be read as a sum of $2x$ and 3 (rather than as a product of 2 and $x + 3$). Teaching students to recognize the gluing operation is a good first step in teaching them to see how order of operations creates the very architecture of an algebraic expression.

Try this: Identify the gluing operation.

(a) $x + 3(x + 2)$

(b) $(x + 3)(x + 2)^2$

DIAGRAMMING EXPRESSIONS AS TREES

Naming the gluing operation is the first step in describing how an algebraic expression is put together. Students then need to be able to identify the pieces that the glue holds together. Frequently, the pieces themselves are algebraic expressions, which, in turn, have their own internal structure in which a gluing operation connects component pieces.

I have found that tree diagrams are a powerful tool for helping students see the multileveled structure of algebraic expressions. (I am certainly not the first to represent algebraic expressions using tree diagrams; see, for example, Sleeman [1984], Ernest [1987], Thompson and Thompson [1987], and Kirshner and Awtry [2004].)

Figure 1 shows, step by step, how to construct a tree for $x + 3(x + 2)$. First, we identify the gluing operation and write it in a circle. In this case, the gluing operation is addition. Next, we draw branches from the gluing operation, one branch for each piece that is glued together. Our example has just two terms, x and $3(x + 2)$, so we draw two branches. The first term, reading from left to right, is just the individual variable x , so we write the x at the bottom of the left branch. The second term, however, is $3(x + 2)$, which is itself an expression with internal structure. To depict it in the diagram, we ask, What is the gluing operation of this expression? In this case, the expression is a multiplication of two pieces (factors), so to include $3(x + 2)$ in our diagram, we draw a multiplication sign in a circle, with two new branches below. Proceeding similarly, we write the individual number 3 beneath the left branch of the multiplication, write another

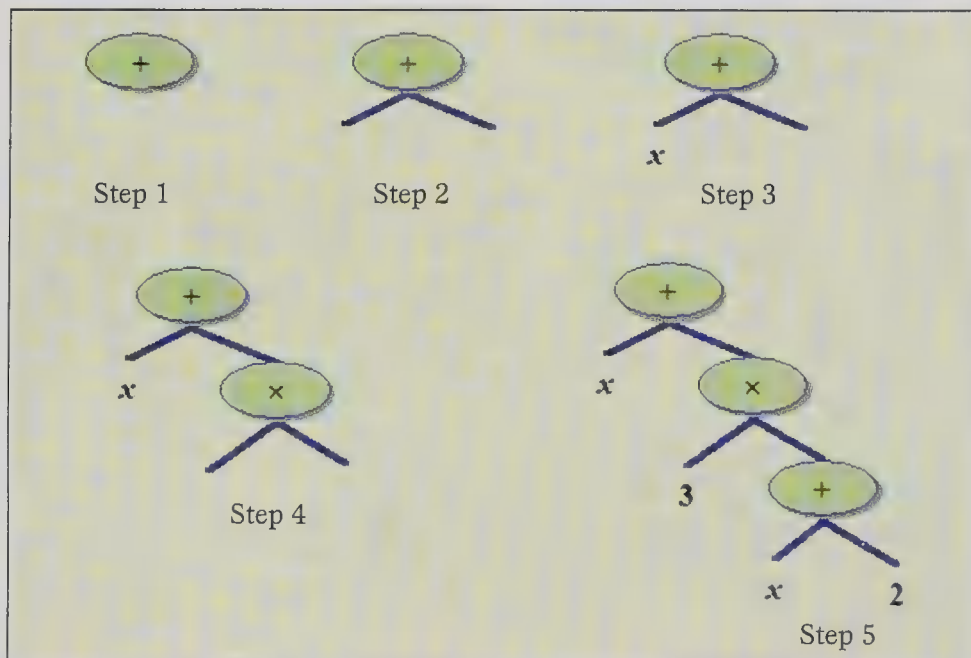


Fig. 1 Identifying the gluing operation is essential to being able to construct an expression tree.

addition sign (with two branches) beneath the right branch, and finally fill in x and 2 beneath those branches, obtaining the final result.

An expression tree uses more vertical space than $x + 3(x + 2)$, the incredibly compact representation in standard algebraic notation. But the tree is far more elegant and less unwieldy than a purely verbal description (“a sum of two terms, the second of which is a product of two factors, the second of which is a sum of two terms”). With just enough detail to avoid ambiguity, students can use a tree diagram to lay bare an expression’s architecture powerfully but economically.

Some of my students, especially those with certain spatial processing difficulties, have struggled to learn to use tree diagrams. However, for students who do learn to draw and interpret tree diagrams, the exercise forces them to confront the fact that the usual representations of algebraic expressions—strings of letters, numbers, operations, and grouping symbols—are merely representations of these expressions. Other ways of writing an expression are possible. When students can represent an expression in multiple ways, they become able to separate the essential features of the expression itself from those that happen to characterize any one representation system.

Try this:

- (a) Draw a tree diagram for $(x + 3)(2x + 5)$.
- (b) Name the expression depicted in **figure 2**.

WHERE DOES A SUBEXPRESSION LIVE?

Once students can illustrate an expression’s structure using tree diagrams, I try to get them to talk about that structure. Students need vocabulary to name parts of an expression according to how they are put together, especially *factor* and *term*.

A *factor* is a piece of a multiplication, whereas a *term* is a piece of an addition. For instance, in $(x + 3)(2x + 5)$, the gluing operation is multiplication, so the primary pieces, $x + 3$ and $2x + 5$, are the expression’s factors. Those factors themselves have internal structure. In the factor $2x + 5$, the gluing operation is addition, so its pieces, $2x$ and 5 , are its terms. Notice, though, that they are not terms of the original expression. The expression $(x + 3)(2x + 5)$ does not have multiple terms because its gluing operation is multiplication. Rather, $2x$ and 5 are terms of the subexpression $2x + 5$, which in turn is a factor of the whole expression.

I use the analogy of a street address. When addressing a letter, we start with the most specific details of the recipient’s location and expand outward from street, to city, to state. The same outward progression applies when describing the

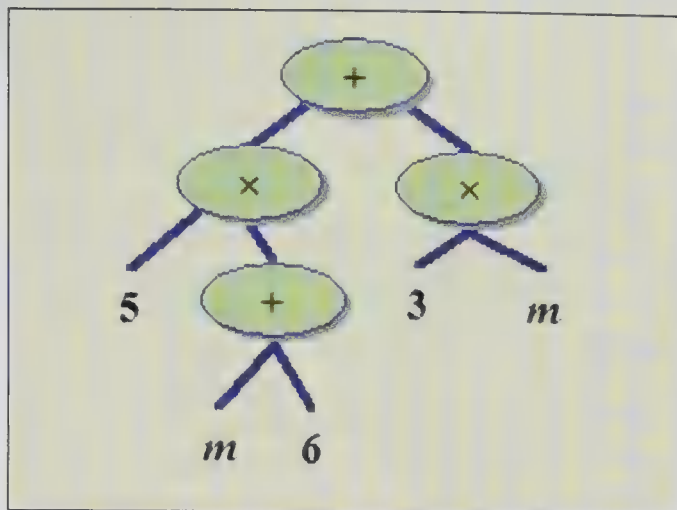


Fig. 2 Can you name the expression shown in this expression tree?

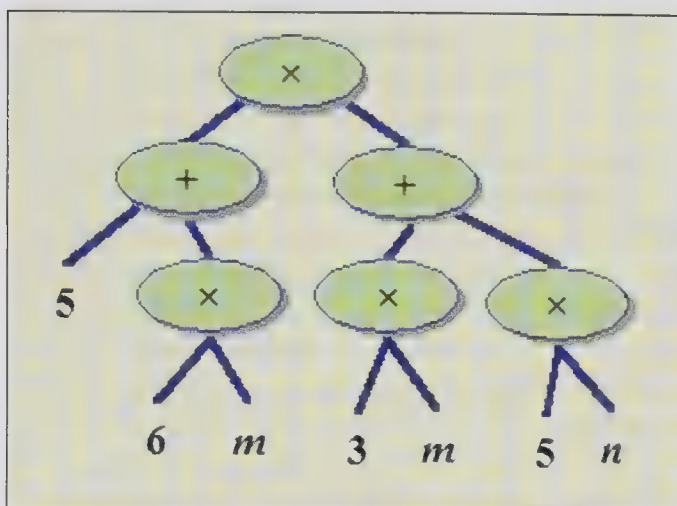


Fig. 3 What is the first factor of the first term of the second factor in this expression?

structural location of a particular subexpression. In $(x + 3)(2x + 5)$, $2x$ is the first term of the second factor of the expression. Similarly, 2 is the first factor of the first term of the second factor.

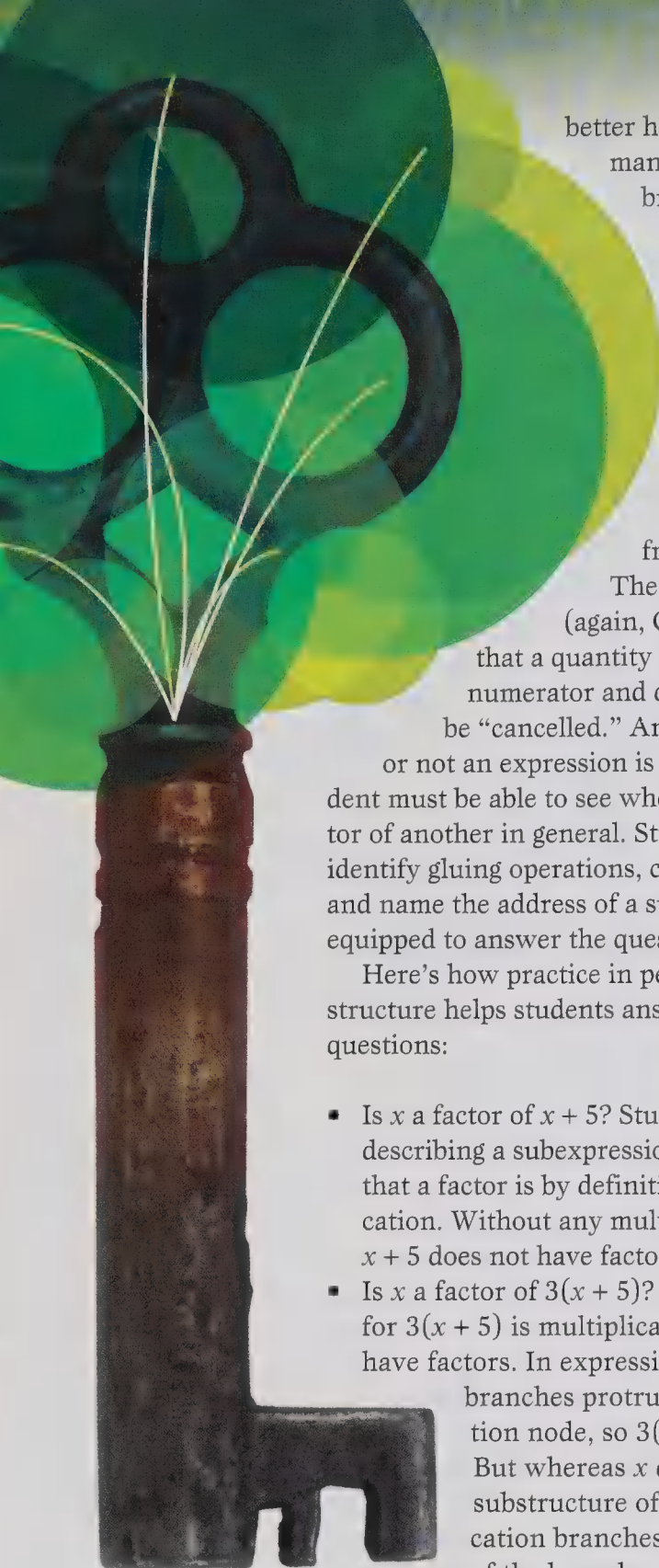
To help students become fluent in this way of describing parts of expressions, I use two types of exercises: I give the address and ask for the specific subexpression, or I give the specific subexpression and ask for the address. For both types of exercise, I might give the expression in either standard or tree diagram notation.

Try this:

- (a) In $2x + 3 + 5(x + 3)(3w + 4)$, what is the address of $3w$?
- (b) What is the first factor of the first term of the second factor in the expression depicted in **figure 3**?

TACKLING “STRANGE MISTAKES” IN SIMPLIFYING FRACTIONS

I introduce these structural concepts when students are learning algebraic fundamentals initially. Then I reference these ideas as needed when teaching later concepts. I have found that many students can



better handle the classic symbol manipulation work of algebra when equipped with tools for analyzing expression structure.

How do these approaches help students avoid making the “strange mistakes” that Grossman noted

when trying to simplify fractions, for example?

The “essence of the matter” (again, Grossman’s phrase) is that a quantity must be a factor of both numerator and denominator in order to be “cancelled.” And to determine whether or not an expression is a factor of each, a student must be able to see whether one thing is a factor of another in general. Students who can already identify gluing operations, create expression trees, and name the address of a subexpression are better equipped to answer the question, Is x a factor of y ?

Here’s how practice in perceiving algebraic structure helps students answer these sorts of questions:

- Is x a factor of $x + 5$? Students with practice in describing a subexpression’s address will know that a factor is by definition a piece of a multiplication. Without any multiplication operation, $x + 5$ does not have factors.
- Is x a factor of $3(x + 5)$? The gluing operation for $3(x + 5)$ is multiplication, so $3(x + 5)$ does have factors. In expression tree notation, two branches protrude from the multiplication node, so $3(x + 5)$ has two factors. But whereas x does occur within the substructure of one of these multiplication branches, x is not the entirety of the branch, so x is not a factor of $3(x + 5)$.
- Is 3 a factor of $3x + 5$? Here, 3 is an entire branch from a multiplication node on the expression tree. However, that multiplication is not the gluing operation for $3x + 5$. Students can name the 3’s address as a factor of the first term, $3x$. But 3 is not a factor of $3x + 5$.

Could exercises about glue and trees and addresses inoculate students against their notorious tendency to cancel incorrectly when simplifying fractions? The common errors that Grossman described result from overgeneralization. Students are apt to cancel anything that appears in both the numerator and denominator, regardless of whether

or not that thing is a common factor of both the entire numerator and the entire denominator. I used to say something like, “You can cross out only something that is being multiplied.” However, that advice does not get at the heart of the matter. My students who followed this advice could still eliminate the 3s in expressions such as

$$\frac{3x + 7}{3y + 8},$$

because the 3s are factors of something (albeit not the entire numerator and denominator).

Equipped with tools for seeing structure, students can confidently determine whether a fraction can be simplified in three steps. Let’s see how this plays out in another example.

Suppose that a student is looking at the algebraic fraction

$$\frac{3x}{4x + 5}.$$

First, I would ask the student to identify any candidates for common factors, and the student would recognize that x occurs in both the numerator and denominator. Second, I would ask the student, “Is x a factor of the numerator?” and the student, experienced in describing subexpression addresses, would check and say yes. Similarly, I would ask the student, “Is x a factor of the denominator?” and the student, perhaps noting that addition, not multiplication, is the gluing operation and is at the top of the denominator’s tree, would say no. Third, I would ask the student if we can “divide away” the x s, and the student would conclude that we cannot, because although x is a factor of the numerator, it is not a factor of the denominator.

For students using this method, the hardest step of the process is remembering to use the process. This series of questions only works to constrain overgeneralization if students remember to ask themselves these questions before simplifying. Therefore, students need many opportunities to practice judging proposed simplifications as correct or incorrect and identifying whether any simplification is possible. Here, too, I have developed my own question types because I have not seen any curriculum materials designed specifically for having students practice constraining their tendency to overapply simplification techniques.

Try these: Each of these expressions contains a 3 in both the numerator and denominator. Simplify by eliminating the 3s or state that the fraction cannot be simplified.

- (a) $\frac{m+3}{n+3}$
- (b) $\frac{3x}{x+3}$
- (c) $\frac{3x}{4(x+3)}$
- (d) $\frac{3x}{3(x+4)}$
- (e) $\frac{3(x+4)+2}{3x}$

TEACHING THE ESSENCE OF THE MATTER

All the exercises discussed here force students to engage with the structure of algebraic expressions and with how that structure permits subsequent transformations. Some teachers might choose to supplement these kinds of exercises with a non-structural approach by having students substitute numbers for each variable. For example, students could establish the nonequivalence of

$$\frac{m+3}{n+3} \text{ and } \frac{m}{n}$$

by substituting various values for the variables and checking whether the expressions are always equal. Substitution can help students establish equivalence because it can ground them in the real numeric equality that algebraic equivalence represents. That said, algebra is powerful because it permits us to work directly on general expressions, a skill that is primarily gained through direct practice. Students must eventually be able to transform algebraic expressions structurally according to the rules of the game—without having to routinely check their work through substitution—to fully benefit from algebra's power.

Some students may find a structural focus challenging. Certainly, it sets apart students who know how to transform an algebraic expression because they understand its structure conceptually from students who have memorized an algebraic transformation by rote or by superficial visual cues. But I have come to believe that teaching algebraic structure explicitly is worth the trouble. A structural approach enforces a deep conceptual understanding on students—at the very conceptual depth that is inherent to the algebra. Any easier way that I might teach this content without involving such conceptual understanding is a memorization approach; my students might emerge with a laundry list of memorized responses to specific stimuli but without the genuine fluency that would allow them to respond flexibly to previously unseen situations.

In any case, once my students have mastered these skills, they are delighted to have become technically proficient enough to truly understand the structure of the algebraic expressions with which they are working. They too would rather understand deeply than memorize arbitrary procedures for various situations just because I told them to.

If we are serious about teaching students to become genuinely fluent in algebra, then our teaching must engage students in algebra at its actual level of conceptual sophistication while being accessible to them as beginners. To accomplish this, we will need to continue developing new exercises and approaches that bring algebraic structure down to earth for our students. Otherwise, we—like Grossman—will have too many students who continue to miss the essence of the matter.

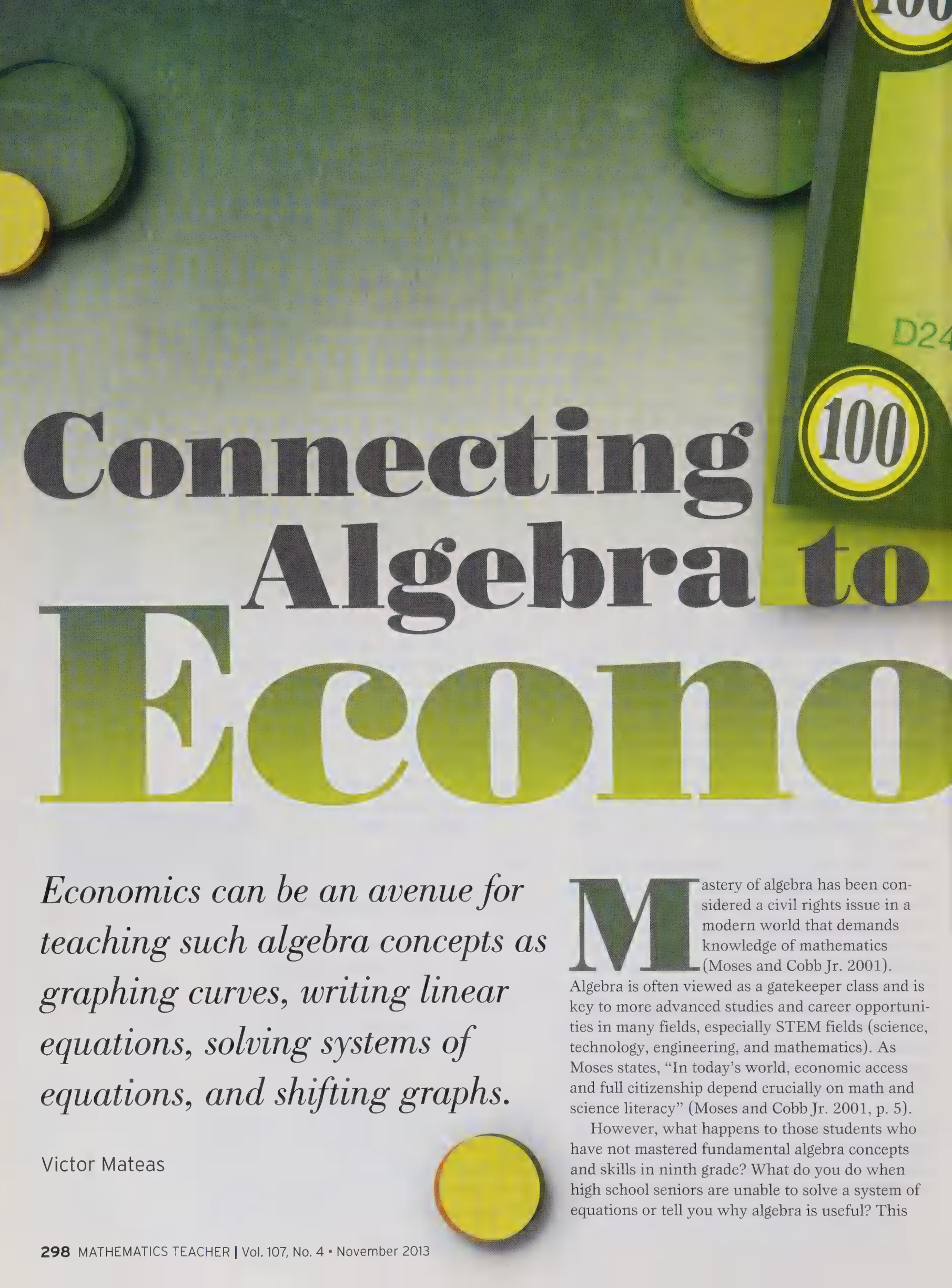
REFERENCES

- Ernest, Paul. 1987. "A Model of the Cognitive Meaning of Mathematical Expressions." *British Journal of Educational Psychology* 57 (3): 343–70. doi:<http://dx.doi.org/10.1111/j.2044-8279.1987.tb00862.x>
- Grossman, August. 1924. "An Analysis of the Teaching of Cancellation in Algebraic Fractions." *Mathematics Teacher* 17 (2): 104–9.
- Kirshner, David, and Thomas Awtry. 2004. "Visual Salience of Algebraic Transformations." *Journal for Research in Mathematics Education* 35 (4): 224–57. doi:<http://dx.doi.org/10.2307/30034809>
- Sleeman, Derek. 1984. "An Attempt to Understand Students' Understanding of Basic Algebra." *Cognitive Science* 8 (4): 387–412. doi:http://dx.doi.org/10.1207/s15516709cog0804_4
- Thompson, Patrick W., and Alba G. Thompson. 1987. "Computer Presentations of Structure in Algebra." In *Proceedings of the Eleventh Annual Meeting of the International Group for the Psychology of Mathematics Education*, edited by Jacques C. Bergeron, Nicholas Herscovics, and Carolyn Kieran, vol. 1, pp. 24–54. Montreal: University of Quebec at Montreal.



ETHAN M. MERLIN, ethanmerlin@gmail.com, has taught middle school and high school mathematics at Sidwell Friends School, Thurgood Marshall Academy Public Charter High School, and Charles E. Smith Jewish Day School, all in the Washington, D.C., area.





Connecting Algebra to Econo

Economics can be an avenue for teaching such algebra concepts as graphing curves, writing linear equations, solving systems of equations, and shifting graphs.

Victor Mateas

Mastery of algebra has been considered a civil rights issue in a modern world that demands knowledge of mathematics (Moses and Cobb Jr. 2001).

Algebra is often viewed as a gatekeeper class and is key to more advanced studies and career opportunities in many fields, especially STEM fields (science, technology, engineering, and mathematics). As Moses states, "In today's world, economic access and full citizenship depend crucially on math and science literacy" (Moses and Cobb Jr. 2001, p. 5).

However, what happens to those students who have not mastered fundamental algebra concepts and skills in ninth grade? What do you do when high school seniors are unable to solve a system of equations or tell you why algebra is useful? This



was the situation I faced as a first-year teacher in the Boston Public Schools, working with high school seniors, many of whom lacked understanding of fundamental algebra concepts. As a teacher with a background in science and engineering, I understood the value and usefulness of mathematics and set out to teach a unit on algebra as it connected to another practical discipline—economics.

The unit lasted approximately three weeks and covered topics such as graphing, using and interpreting tables and graphs, writing equations for lines, solving systems of equations, and writing equations for shifted graphs. These lessons were taught to twelfth grade students with special needs in a substantially separate mathematics class that met five times a week for fifty-five minutes. The unit was designed for students who had previously

struggled in mathematics, but the ideas presented here and the connections between algebra and economics can be used in any beginning algebra course.

WHY ECONOMICS?

Economics is a field that relies heavily on algebra, making it easy to find useful and concrete connections to the mathematics taught in schools. Further, economics is a discipline that is immediately relevant to the lives of students, and examples can be chosen that connect to student interests and experiences. Last, exposing students to a potential college major and incorporating college-level text into a unit will better prepare them for the academic opportunities and rigor of postsecondary education.

Are Supply and Demand Curves Really Straight Lines?

In this unit, we assumed that all supply-and-demand curves are linear. However, supply-and-demand curves are often nonlinear. Many introductory economics texts use linear curves, however, because they simplify calculations while maintaining the inverse and direct relationships in the laws of demand and supply, respectively (Rittenberg and Tregarthen 2009; Scott 1997). To understand why a demand curve might be nonlinear, consider the following scenarios:

- Scenario A:* A bakery decreases the price of cupcakes from \$10 to \$8. The quantity demanded then increases by 5 cupcakes.
- Scenario B:* A bakery decreases the price of cupcakes from \$5 to \$3. The quantity demanded then increases by 20 cupcakes.

In both scenarios, there is a \$2 price reduction. So why would the change in quantity demanded be different?

The quantity demanded did not change proportionally because the price reduction took place at different price levels. In scenario A, the \$2 drop occurred when prices were already high; few buyers would consider this decrease substantial. In scenario B, however, the \$2 drop occurred at a lower price range; as a result, more people found the price decrease significant. In economics, the responsiveness of quantity demanded to price changes is called the price elasticity of demand. Similarly, the responsiveness of quantity supplied to price changes is called the price elasticity of supply (Tucker 2005).

THE LAWS OF SUPPLY AND DEMAND

The first few lessons in this unit taught students the laws of supply and demand and how to represent those laws using tables and graphs. We began with a class discussion probing students' prior knowledge of economics. Many students, on the basis of their exposure to the news, associate the term *economics* with money, jobs, and unemployment. After discussing our experiences with the term, we defined *economics* as the study of how people choose to use limited resources for the production of goods and services (Tucker 2005).

After defining the term, the students observed the law of demand in action using a concrete example they could get excited about. Students were asked how many of my famous carrot-cake cupcakes they would buy at various prices. I started by polling them on how many cupcakes they would buy if I sold cupcakes at \$1 apiece, then \$2, and so on up to \$10. By adding the quantity demanded by each student to get a total quantity demanded by the whole class, we were able to compare that with the process of summing individual demand to calculate market demand.

We then tabulated the results comparing price to (total) quantity demanded, called the *demand schedule*, and graphed the data to obtain the demand curve (see **fig. 1** for an example of a linear demand schedule and curve; see also the **sidebar**). During this process, we also reviewed the purpose of graphs, the vocabulary associated with graphing, the importance of labeling, and how to choose an appropriate scale. Because students were graphing data that they had generated, they were more invested in its presentation. When analyzing the graph and the table, students saw for themselves an inverse relationship between price and quantity

Demand Schedule for Cupcakes	
Price (\$)	Quantity Demanded (thousands)
7	1
5	4
3	7
1	10

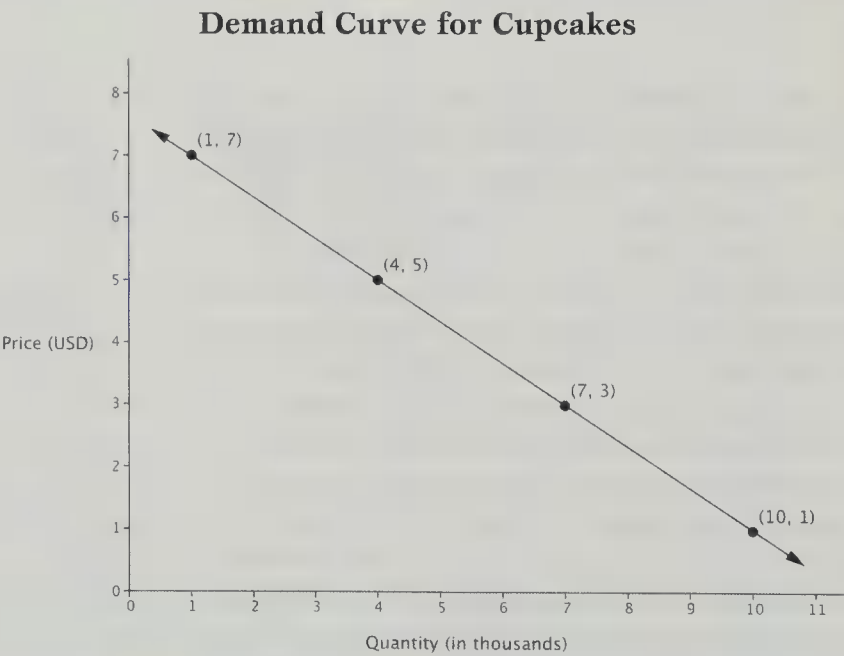


Fig. 1 The data in the demand schedule can be graphed as a demand curve.

demanded—in other words, the law of demand. After the law of demand was demonstrated, students were asked to graph demand curves given linear demand schedules and to interpret those graphs to determine how much of a given product would be demanded at various price levels and vice versa.

Once students understood the law of demand, we moved on to the law of supply. This law depends on the supplier’s willingness to produce a given product; thus, it is conceptually more difficult for students to understand. Continuing the cupcake example, I created two new scenarios. In scenario 1, I was selling cupcakes to students and colleagues in my school. However, people were willing to spend only \$1 apiece. In scenario 2, I had been invited to appear on *The Martha Stewart Show* to bake my cupcakes. People were now willing to spend as much as \$10 a cupcake. I asked students in which scenario they thought I would spend my time and resources baking a large number of cupcakes. They all said scenario 2, and we discussed the direct relationship between price and quantity supplied—the law of supply. Once again, students were asked to graph supply curves given a supply schedule and interpret those graphs for various prices and quantities supplied.

CALCULATING MARKET EQUILIBRIUM

Once students had a solid understanding of supply and demand and how to graph these relationships from a table, they learned in the next few lessons how to model the laws using linear equations and how to use the mathematical models to calculate market equilibrium.

This part of the unit began by having students graph both supply and demand curves on the same set of axes. Students noticed that the two lines intersected, and they were asked a series of questions to help them interpret the graphs and the significance of the point of intersection. For prices above the point of intersection, students saw that the quantity demanded (Q_D) was less than the quantity supplied (Q_S); in other words, there was a surplus of product. For prices below the point of intersection, students saw that the quantity demanded was higher than the quantity supplied; in other words, there was a shortage of product.

It was only at the price level where the two lines crossed that the quantity demanded was equal to the quantity supplied and market equilibrium existed (see **fig. 2**). Having described the various market conditions, students were asked to think about how (and why) suppliers would react to the three scenarios presented in **figure 2**.

Students initially used only graphs to find the price and quantity at market equilibrium, but they quickly realized that an easier method for calculating the market equilibrium would be helpful. At this

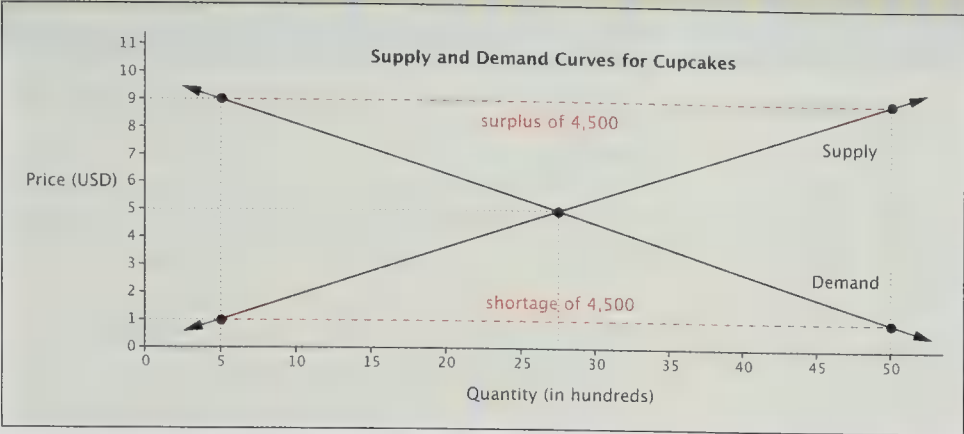


Fig. 2 When the price is \$9 each, $Q_D < Q_S$, so there is a surplus of 4,500 cupcakes. When the price is \$1 each, $Q_D > Q_S$, so there is a shortage of 4,500 cupcakes. When the price is \$5 each, $Q_D = Q_S$, so there is market equilibrium.

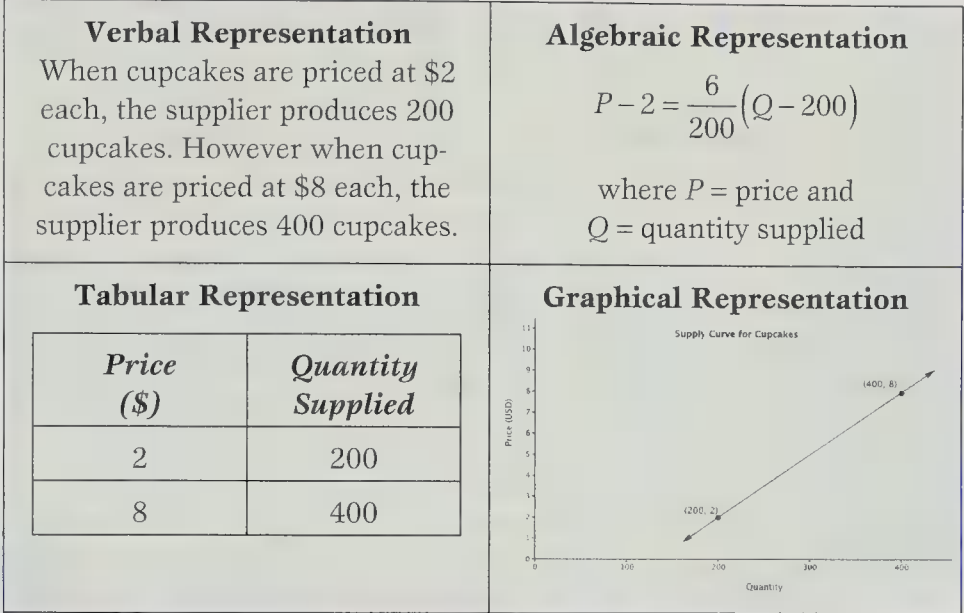


Fig. 3 Supply can be represented in multiple ways.

point, students were taught the point-slope form of a line as a modified version of the slope formula, which they already knew. Students were then asked to write equations for supply and demand curves given word problems, tables, and graphs. Thus, students had the opportunity to see verbal descriptions, tables, graphs, and equations as multiple representations of the same real-world scenario (see **fig. 3**). Once they were able to write equations for a supply or a demand curve, students could solve the linear equation and determine exact values for the quantity supplied or demanded at different prices.

Although solving linear equations is useful for calculating the value of a variable, students also explored the limitations of algebraic models. Students found that even though linear equations represented lines that extended infinitely in both directions, some values of the domain and range, such as negative values for price or quantity supplied or quantity demanded, had no meaning in the real world. Therefore, although mathematical models have many benefits, such as describing real-world situations and providing a mathematical structure that can be manipulated and analyzed, students

Table 1 Nonprice Determinants of Demand and Supply		
Nonprice Determinant of Demand	Relationship with Demand	Example
Number of buyers	Direct	An increase in the number of students with a cold will increase the demand for Kleenex.
Tastes and preferences	Direct	A decrease in the popularity of Silly Bandz will decrease the demand for Silly Bandz.
Income	Direct or Inverse	As a family's income increases, the demand for organic produce increases, and the demand for generic products decreases.
Nonprice Determinant of Supply	Relationship with Supply	Example
Number of sellers	Direct	An increase in the number of iPhone® app developers will increase the supply of apps.
Technology	Direct	With the invention of the cotton gin, there was an increase in the supply of cotton.
Resource prices	Inverse	A decrease in the price of silk will increase the supply of silk clothing.

Source: Tucker (2005)

Scenario: Consider the demand curve for Silly Bandz, D_1 , while the toy is popular. However, after the fad passes, Silly Bandz now sell at 30,000 units less than before given any price level.

Prompt: Write the equation for both demand curves shown.

$$D_1: P - 2 = -\frac{1}{20}(Q - 50)$$

$$D_2: P - 2 = -\frac{1}{20}(Q - 20)$$

Question: What term could you add to the equation for D_1 to get the equation for D_2 ? How does this term relate to the way the graph shifted?

$$P - 2 = -\frac{1}{20}(Q - 50 + 30) \rightarrow$$

$$P - 2 = -\frac{1}{20}(Q - 20)$$

A +30 term was added to Q when the graph shifted 30 to the left.

Fig. 4 Students need to show the effect of nonprice determinants.

saw the importance of questioning when a model is viable and when it is not.

Finally, students were taught how to calculate the market equilibrium using a system of equations that they wrote for the supply and demand curves. Students solved by using both substitution and elimination, an alternative approach to solving graphically. The ease with which market equilibrium could be determined using equations, in certain situations, provided students with a concrete advantage for using algebraic models.

SHIFTS IN THE MARKET

Up to this point in the unit, only two variables had been discussed—price and quantity supplied or demanded. Price is a major factor in determining the quantity that will be supplied and demanded, but it is not the only factor. Economists describe a range of other factors that influence the supply and demand; these are known as nonprice determinants (see **table 1** for selected factors). Whereas increases and decreases in price cause movement on the curve, a change in a nonprice determinant will cause the entire demand or supply curve to shift. The reason for this is that a nonprice determinant affects how a product will be demanded or supplied at every price level; therefore, every point in a curve is affected.

For this set of lessons, students read and took notes on the section in *Microeconomics for Today* (Tucker 2005) describing nonprice determinants of supply and demand. As a class, we then discussed various scenarios in which these factors could come into play, and students made qualitative statements about how the supply or demand for a product would change (see examples in **table 1**).

Next, students were asked to think about the effect that nonprice determinants would have on the graphs of supply and demand—namely, that there is a shift in the curve because a third variable is introduced. By looking at supply and demand curves before and after a nonprice determinant changes, we analyzed how the quantity changed for a given price (see **fig. 4**).

Students also looked at a given quantity and saw how the price changed because of the nonprice determinant. Helping to identify the amount the graph shifts vertically (by looking at price change) or horizontally (by looking at quantity change) led us to discuss how the equation of the shifted line compared with the equation of the original line (see **fig. 4**).

Finally, students were asked to write their own equations for shifted supply and demand curves given (1) sufficient information to write an equation for the curve before the nonprice determinant changes and (2) a description of how the quantity supplied or demanded changed at every price level as a result of the nonprice determinant change. Alternatively, students were also given a description of how price changed at every quantity level.

REAL-WORLD APPLICATION: iPHONES THEN AND NOW

To conclude this unit, students worked on a final project (available at www.nctm.org/mt) that

compared the iPhone® market in 2008 and 2010. In 2008, iPhones were one of the few smartphones available and were unmatched in the features and number of apps offered. By 2010, however, iPhones had increased competition; Android™ smartphones were introduced, and new BlackBerry® models were on the market.

Given supply and demand schedules for iPhones in 2008, students were asked to graph the supply and demand curves, write equations for those curves, and then use algebraic techniques to determine the market equilibrium (see **fig. 5**). Second, students were given information on how the demand for iPhones changed from 2008 to 2010. Using this information, students had to graph the new demand curve (see **fig. 5a**), write an equation for the demand in 2010, and calculate the market equilibrium for iPhones in 2010.

Students were also asked questions to help them understand why the market equilibrium of iPhones changed between 2008 and 2010. They were asked to identify the type of nonprice determinant of supply and demand that came into effect when other smartphones appeared on the market (i.e., the price of substitute goods) as well as to describe how that nonprice determinant influenced the supply and demand of iPhones.

To further understand the need for price changes based on changes in the market, students

Annual Perspectives in Mathematics Education

2015 Call for Chapters Announcement

can be found at www.nctm.org/APME2015. The editorial panel will seek chapters that bridge research and practice and that highlight important issues related to assessment as it informs teaching and learning for all learners pre-K–12. Intention to submit forms, which are available at www.nctm.org/publications, will be due in March 2014 and full chapter drafts in May 2014.

The 2015 volume of NCTM's newest annual publication, **Annual Perspectives in Mathematics Education** (APME), which highlights current issues from multiple perspectives, will focus on **Assessment to Enhance Learning and Teaching**.

The full call for manuscripts for the 2015 APME, with details regarding suggested topics and submission dates,

Coming in April 2014... Watch for the inaugural volume of *Annual Perspectives in Mathematics Education* (APME) with a focus on **Using Research to Improve Instruction**.

You can order your copy from the catalog at www.nctm.org/catalog or be the first to get one when it comes out by checking the automatic-order box on the NCTM membership or renewal forms.

**Please consider contributing to
or purchasing these important
publications.**

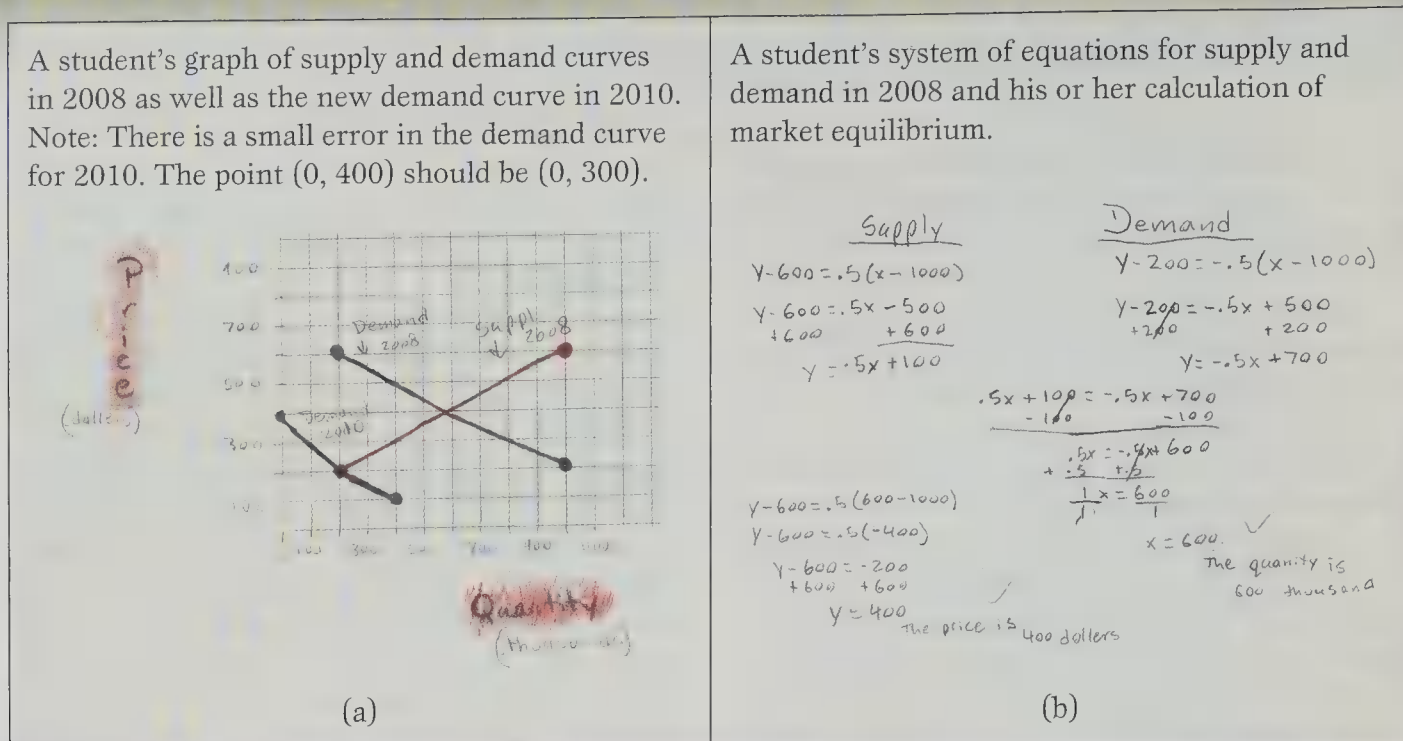


Fig. 5 Student work ties together several representations.

were asked to consider a hypothetical situation in which Apple® priced iPhones in 2010 the same way that they did in 2008. Using their algebraic models or graphical representations, students were able to see that the quantity of iPhones demanded would be zero in the scenario based on the demand in 2010. In fact, if we use the algebraic equation, the quantity demanded would be a negative value, which can be interpreted as no demand.

Finally, the project required students not only to consider how and why market equilibrium for a product changes but also to make an educated prediction as to how the market might change in the future. Students were asked to consider a hypothetical situation in which Apple strikes a deal with a microchip producer, thereby reducing the cost to produce iPhones. Students then needed to use their knowledge of economics and the mathematical models that they learned in class to predict how market equilibrium would change given these new conditions.

POSITIVE RESULTS

This unit combining algebra with economics was well received. Most students were engaged, and their algebra skills improved significantly. Students were also excited about being held to high expectations and challenged with college-level material; as one student commented, “[This is important] because it’s getting you ready for college.” Some even expressed interest in majoring in economics, asking questions about the field that went beyond the topics discussed in class.

Our mission as teachers is not only to teach the content but also to inspire our students by exposing them to new opportunities and ideas. One excellent way of accomplishing this is by showing the many ways in which mathematics can be connected to other

disciplines and to the real world. These connections are important to make and will benefit students, regardless of their mathematical skills. I hope that the ideas presented here provide some assistance to teachers in the way they approach algebra instruction.

REFERENCES

- Moses, Robert P., and Charles E. Cobb Jr. 2001. *Radi-cal Equations: Civil Rights from Mississippi to the Algebra Project*. Boston: Beacon Press.
- Rittenberg, Libby, and Timothy Tregarthen. 2009. *Principles of Economics*, v.2.0. Irvington, NY: Flat World Knowledge. <http://catalog.flatworldknowledge.com/catalog/editions/10>
- Scott, Carole E. 1997. “Identifying the Profit-Maximizing Price May Be Much Tougher Than Textbooks Indicate.” *Business Quest*. <http://www.westga.edu/~bquest/1997/profit.html>
- Tucker, Irvin B. 2005. *Microeconomics for Today*. 4th ed. Mason, OH: South-Western.



VICTOR MATEAS, vmateas@edc.org, taught high school mathematics in the Boston Public Schools before joining the Education Development Center in Waltham, Massachusetts, as a research associate. He is interested in mathematical practices and applications of mathematics.

more4U

This activity sheet is available to teachers as a Word document that can be copied and edited for classroom use; go to www.nctm.org/mt.

All calls are open calls except for the 2015 Focus Issue.



Reasoning and Sense Making

Describe how you generate a culture of reasoning and sense making in the classroom.



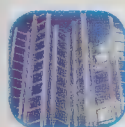
Open the Door and Keep It Open

Assist teachers working with students who are in first-year algebra or geometry courses or in the first years of an integrated curriculum.



Mathematizing the World: An Invitation to Modeling

The increased emphasis on modeling in the Common Core State Standards for Mathematics makes this call timely.



Great Problems

Share your great problems with other teachers.



Statistics and Probability

How does the recent increased attention to data and chance affect your teaching and your students' learning?



On the Front Burner: Emerging Issues in Mathematics Education

Stir the pot on provocative and contemporary issues in mathematics education.



2015 Focus Issue: Creating Classroom Communities

Help *MT* readers learn ways to capitalize on the strengths that cultural, racial and linguistic diversity bring to our classrooms and schools. Submit by **May 1, 2014**.

The following departments also seek manuscripts: Activities for Students, Connecting Research to Teaching, Media Clips, Technology Tips, and The Back Page: My Favorite Lesson. Calendar problems are always welcome.

For more information, go to <http://www.nctm.org/publications/content.aspx?id=8756>.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

MATHEMATICS
teacher



CALL FOR MANUSCRIPTS

Linked Representations in Algebra: Developing Symbolic Meaning

The use of symbols provides mathematics with enormous power. From the learner's perspective, however, symbols can present enormous obstacles. Using symbols to encapsulate big ideas allows readers to see various stages of arguments within one field of view. However, for many learners, unpacking symbolization poses a significant hurdle in the development of conceptual understanding. Here we compare and contrast symbolic reasoning approaches that algebra students used when solving equations.

What is a root of an equation, and how is it related to various representations? More generally, what does it mean for a value to be a solution to an equation? These are standard questions that we expect students to be able to address and discuss. Although many students may be able to solve equations, far too many have limited conceptual understanding and rely primarily on procedural knowledge of the equation-solving process.

AARON'S SOLUTION TO A QUADRATIC EQUATION

Consider the following situation in which a student, Aaron, tried to solve

the quadratic equation below during an interview:

$$(x - 2)(x + 3) = 6 \text{ (no graph given)}$$

Aaron began by setting each factor equal to 6 (see **fig. 1**). Here he applied a memorized procedure that he overgeneralized from previous experiences of solving quadratic equations by setting each factor equal to zero. After finding solutions, Aaron said, "I don't know if those approaches are right . . . I'd have to know exactly where we are at," indicating that he was unsure of his process. When questioned further, he stated that his process might change depending on "if we're trying to find where it's going to cross on the x -axis." In elaborating, Aaron stated that if he were trying to find points on a graph, then he would view the process differently, and he referred to another problem that included a graph along with an equation (see **fig. 2**). However, when he described how his process would change, he repeated the same algebraic steps and changed only the number that he would set each factor equal to (4 instead of 6) (see **fig. 3**).

As Aaron continued to solve the new problem, he noticed an inconsistency:

Connecting Research to Teaching appears in alternate issues of *Mathematics Teacher* and brings research insights and findings to the journal's readers. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more information on the department and guidelines for submitting a manuscript, please visit <http://www.nctm.org/publications/content.aspx?id=10440#connecting>.

Edited by **Margaret Kinzel**, mkinzel@boise.state.edu

Boise State University, Boise, ID

Laurie Cavey, lauriecavey@boisestate.edu

Boise State University, Boise, ID

$$\begin{array}{l} x-2=6 \\ +2 \\ \hline x=8 \end{array} \quad \begin{array}{l} x+3=6 \\ -3 \quad -3 \\ \hline x=3 \end{array}$$

Fig. 1 Aaron shows his algebraic approach for solving $(x-2)(x+3)=6$, setting each factor equal to 6.

Aaron: Uh-oh, something's not right . . . the solutions [don't] match . . . [pressed to explain] [I] got 5 and 2, and I would expect them to match -2 and 1 [pointing to the x -intercepts on the graph]. Um, just because that's where they cross.

At first, it appeared that Aaron had caught his conceptual mistake, but he had simply replaced an algebraic rule with a graphical one (looking for the points where the graph crosses the x -axis). Aaron continued to try to resolve his algebraic solutions with his expectations that these solutions should correspond to the x -intercepts from the graph, but he was unsuccessful and gave up.

Why was Aaron unable to see the error in his solution process even though he did attempt to use multiple representations when a graph was provided? The teacher had made it a point to use multiple representations in solving equations during class, so it is not surprising that Aaron would seek confirmation when the graph was present. However, the use of multiple representations alone is not enough to help students make connections; students need to see how these representations connect.

OUR STUDY WITH COLLEGE ALGEBRA STUDENTS

Our proposal for curricular enhancement is based on a study, conducted over two semesters with college algebra students, that examined how students developed algebraic concepts using various types of technology. During the fall semester, fifteen students participated in the study; four of them also took part in two interview sessions conducted during the last three weeks of the course. During the spring semester, thirty-four students participated in the study; seven of them took part in similar interview sessions. Data were collected during

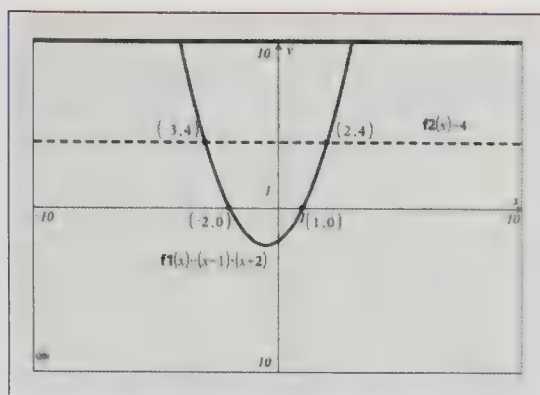


Fig. 2 Given the graph of the function as shown, students were asked to find all solutions to $(x-1)(x+2)=4$ and justify their answer.

both semesters through interviews, researcher field notes, and student artifacts. Because the content of the typical college algebra course is roughly the same as that of a high school second-year algebra course, our results maintain some applicability to the secondary school curriculum.

In this particular study, students in the first semester were taught with standard graphing calculators (the TI-84, which does not allow dynamically linked representations) along with occasional demonstrations by the instructor using a platform that allows dynamically linked representations (TI-Nspire CAS). By *dynamically linked*, we mean representations that update in real time as each is manipulated. During the second semester, each student was issued a TI-Nspire CAS to use during the entire semester, and the same teacher served as a control for instructor influence with respect to general approaches to instruction. The teacher for both semesters was an expert on the use of technology in the teaching and learning of mathematics who has delivered many hours of in-service professional development.

Aaron participated in the first semester of the study. As a result, he was not able to see a real-time linkage between the key features of the graph and their corresponding algebraic counterparts in the symbolic world.

DEVELOPMENT OF SYMBOLIC MEANING

The theoretical framework of this study is based on research with microcomputer-based laboratory (MBL) technology (Lapp and Cyrus 2000) as well as Kaput, Carraher, and Blanton's (2008) model

$$\begin{array}{l} x-1=4 \\ +1 \\ \hline x=5 \end{array} \quad \begin{array}{l} x+2=4 \\ -2 \quad -2 \\ \hline x=2 \end{array}$$

Fig. 3 Aaron's algebraic approach to solving $(x-1)(x+2)=4$ mirrored his algebraic approach to solving $(x-2)(x+3)=6$.

for the development of symbol systems. In the research with MBL, we have seen that real-time changes in linked representations that are simultaneously visible help students make connections among corresponding features of the various representations (Lapp and Cyrus 2000; Beichner 1990; Brasell 1987).

Linking representations deepens understanding of mathematical concepts, but we still must ask how the use of symbol systems influences our learning and, further, our development of new ideas. Kaput, Blanton, and Moreno (2008) propose a model that describes how we develop our understanding of existing ideas as well as how we create new concepts. The key to this model is the communication and analysis between two worlds. One is the "real world" (i.e., the world of broader mathematical or physical experiences), and the other is the world of the symbols that we use to represent these real-world experiences. In this model, the learner initially creates raw representations from experiences; in turn, these representations and understanding of the real world evolve as the student interacts with both worlds. The evolution of the student's view of the real world is relative to the symbolic worlds of B and C , which are represented by A_B and A_C (see **figs. 4a** and **4b**). In this sense, Kaput, Blanton, and Moreno's model is consistent with Tall and Vinner's (1981) view of an evolving "concept image."

This process, although focusing on representations, is consistent with Sfard's (1991) process of interiorization, condensation, and reification, in which concepts become internalized through interaction with the various natures of the mathematical ideas. In some ways, the use of technology here reduces the length of time for Sfard's interiorization and facilitates the condensation process. Alleviating the computational load can

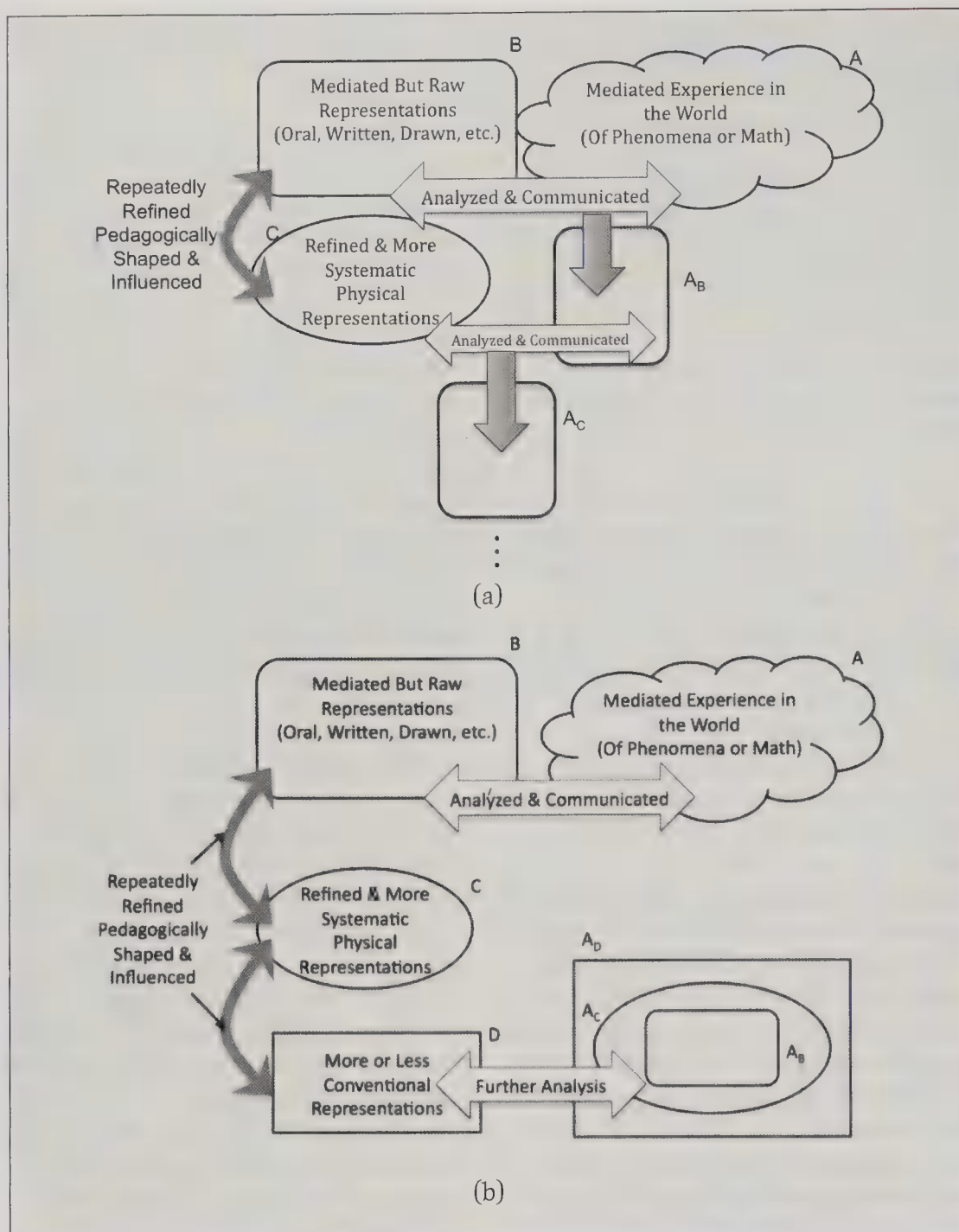


Fig. 4 Kaput, Blanton, and Moreno (2008, pp. 30 and 31) describe a model that explains how symbolic meaning evolves (a) and how symbol systems are nested as they evolve (b).

also keep the lack of procedural fluency from becoming a cognitive obstacle to reaching condensation or reification, as observed by Lapp, Nyman, and Berry (2010).

Although Aaron had access to the use of multiple representations and the teacher routinely used graphical, numerical, and algebraic representations, they were not dynamically linked. Thus, Aaron may have had more difficulty making a connection between the meaning of the graphical representation of x -intercepts and the zero property of multiplication as a technique for equation solving. It is also important to note that Aaron's approach was typical of all

students interviewed during the first semester of the study.

JON'S APPROACH

How do we create experiences that allow students to move fluidly among various representations while maintaining an isomorphic mapping of meaning between various worlds used to represent the ideas involved? One way is to use technology that allows for linked representations.

In contrast to the students interviewed in the fall semester, who used only the basic graphing technology, all the students interviewed in the spring semester showed an ability to move among representations

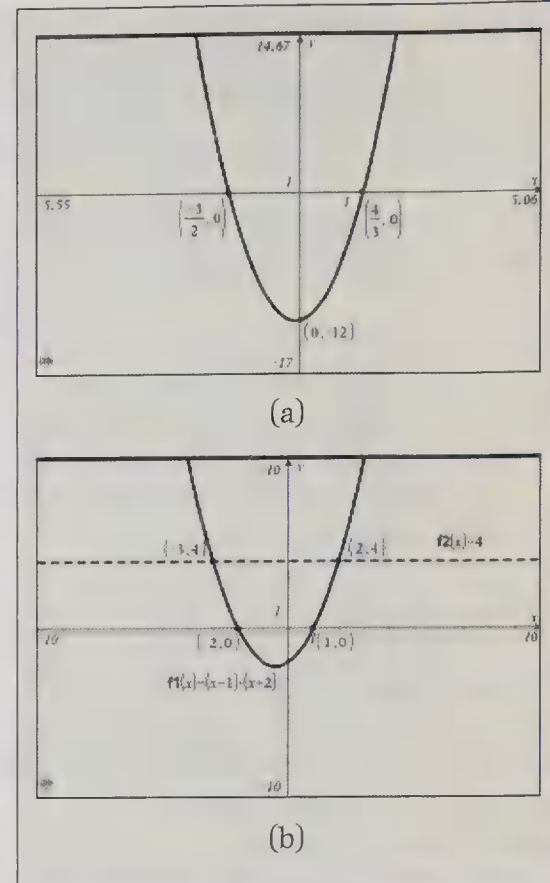


Fig. 5 Jon was able to use his solutions to these problems to help him solve $(x-2)(x+3)=6$. These are the graphs for problems 6 and 7 in Jon's earlier interview.

as a means of justifying their reasoning. To illustrate, we examine Jon's response to the same question posed to Aaron—solving the equation $(x-2)(x+3)=6$. Unlike students from the previous semester, Jon realized that he must get the equation set equal to zero to make the comparison to zeros on the graph of a function. In the following exchange, Jon referred to his solutions to other problems (problems 6 and 7; see fig. 5).

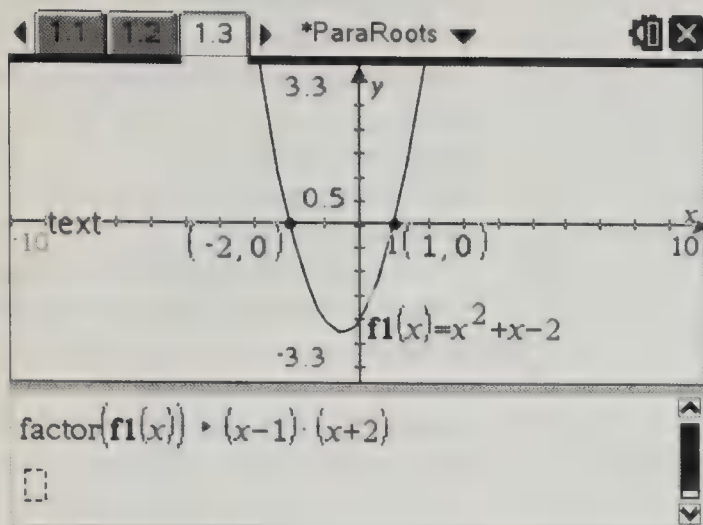
Interviewer: So what do you think is different about this? Why is it important that you get the 6 to the other side?

Jon: Because I think if you were to graph it, you're trying to find a completely different line, if it's a 6 or 0 and if it's 0 . . . You're trying to find y . And then if y was 0 . . . I don't know.

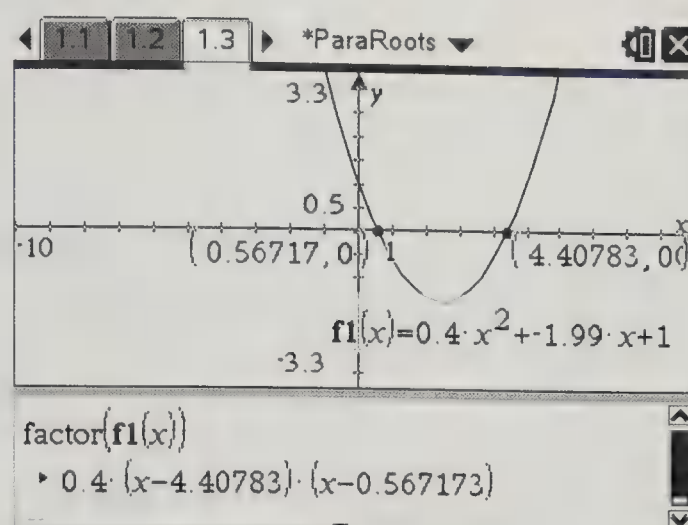
Interviewer: So you're thinking of it from a graphical standpoint?

Jon: Yeah, I'm thinking about it from a graph. If it's equal to 6, it's going to be up higher and it's gonna be . . . You're not looking for the zeros [points to problems 6 and 7 from the earlier questions].

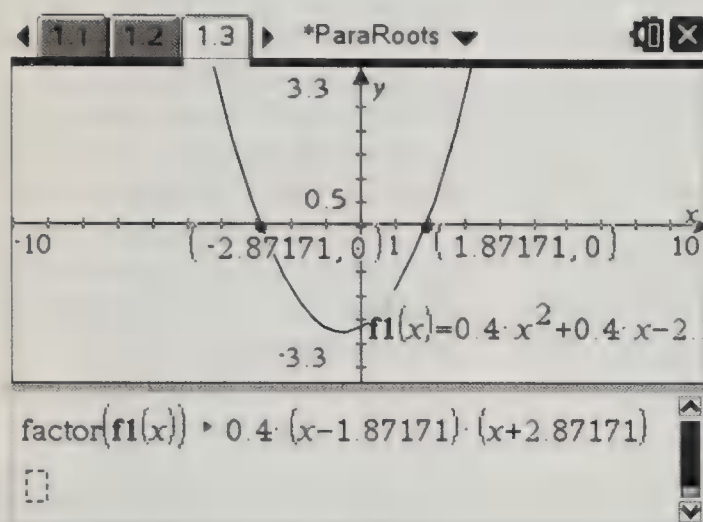
Interviewer: So kind of like the



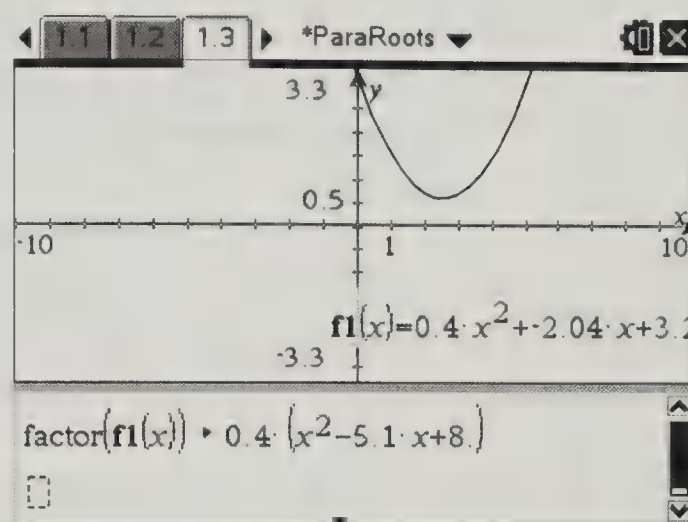
(a)



(c)



(b)



(d)

Fig. 6 These are still images from the dynamic experience of moving a graph and seeing the factored form of the function change in real time. Manipulating the graph enabled Jon to see the relationships between roots and x-intercepts.

difference in those earlier problems?

Jon: Basically, I took number 7 and tried to turn it more into, like, number 6. Like, I tried . . . I just moved it down, so that way I would be able to solve for it. I visualized it as more like bringing it down, and I just solved for the vertex, which I know how to do. When it comes to that, it's more complicated to get to that point.

Interviewer: Oh, the x-intercepts you mean, as opposed to the vertex?

Jon: Yeah, 'cause when it's at equals 6, it's like the whole thing got raised 6. So I just made it . . . I brought it down to the vertex . . . and then I just solved for it there because that's easier for me to do algebraically.

Jon had first been exposed to dynamically linked representations through

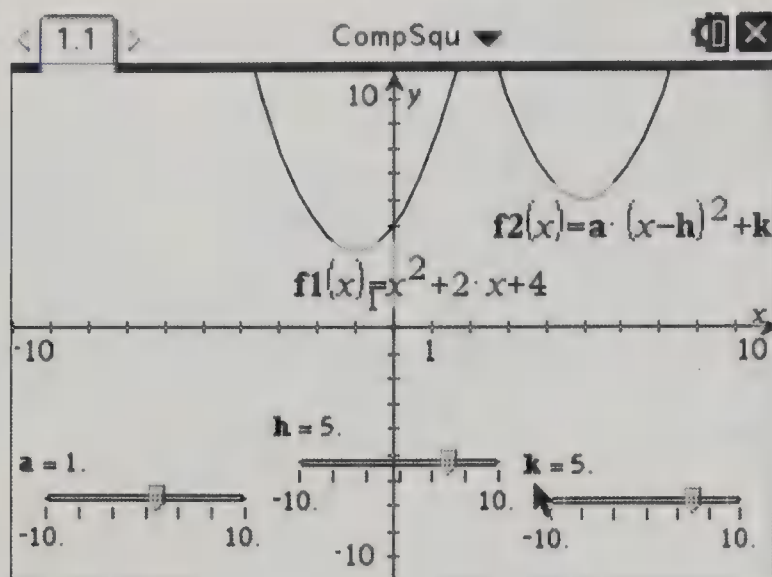
technology-rich investigations in class.

In one investigation, manipulating a graph gave real-time change in the algebraic representation of the quadratic function as well as the factored form of the function. Jon was asked to label the x-intercepts on the graph, and, as he manipulated the graph, he noticed a connection between the numerical values of the x-coordinates of the x-intercepts and the numerical values, r_1 and r_2 , found in the factored form of the function $a(x - r_1)(x - r_2)$ (see **fig. 6**).

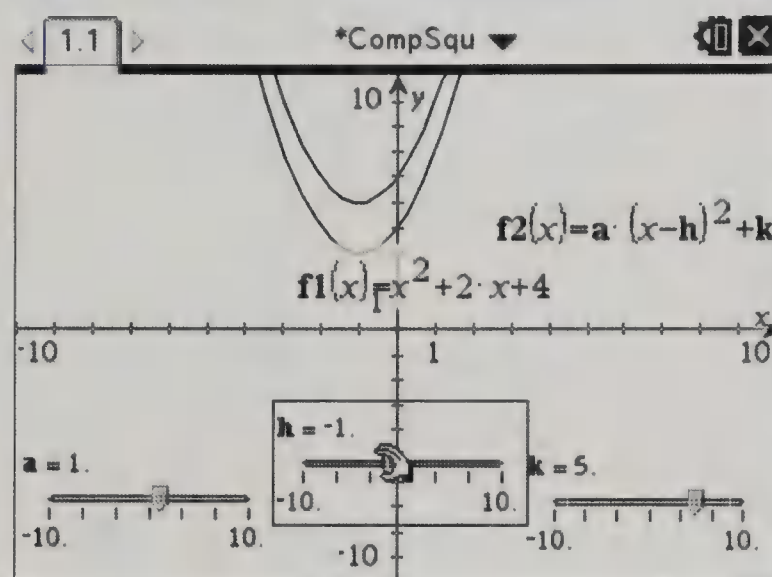
In the past, we have seen that students tend to focus on aspects of a situation that are invariant across representations (Lapp 1997). In this case, the student noticed that, no matter how the graph was manipulated, the numerical values of the x-coordinates in both the labeled x-intercepts and the r_1 and r_2 values

in the factored form of the function remained the same. The classroom investigation also included questions designed to focus the student's attention on the effects on the function's output for entering each x-coordinate of the zeros into the factored form of the function as well as the values of each factor. Our research suggests that this combination—communication and analysis between symbolic worlds—enabled the student to articulate the reason for the use of factoring as a technique for equation solving. For this reason, it is imperative that the equation is set equal to zero before factoring.

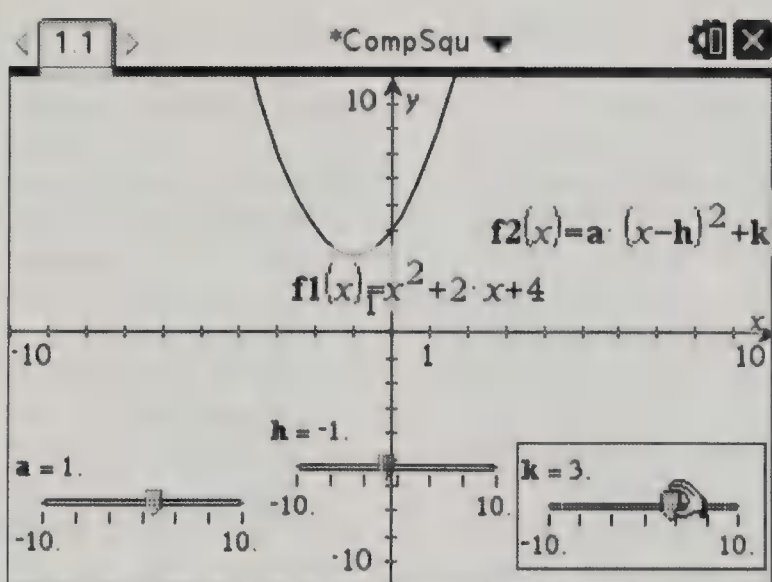
Students in both semesters were exposed to the zero property for multiplication as a reason for solution by factoring, but students in the second semester, using CAS, were allowed to interact with linked representations and



(a)



(b)



(c)

Fig. 7 Students use sliders to manipulate a parabola given in vertex form and see its equivalent expression in standard form. In this way, they were able to make more connections when the quadratic was expressed in vertex form.

were required to justify their reasoning in a lab experience. Students in the first semester were told by the teacher about this reason but simply watched the teacher point out the fact on a single graphical representation that was not linked algebraically.

In another investigation, Completing the Square (see **fig. 7**), students developed a connection between algebraic and graphical representations for the process of completing the square and observed relationships among various parameters found in the algebraic expressions. Here students further experienced the connection between algebraic and graphical transformations.

This experience likely led to Jon's strategy of transforming the graph of a parabola and its intersection with a horizontal line above the x -axis into one in which the parabola was shifted down until the horizontal line was superimposed on the x -axis. In his desire to transform an equation by subtracting 6 from both sides, Jon expressed a graphical understanding linked to the algebraic transformation of subtracting 6. He stated that he was essentially moving the dotted line (see **fig. 5**) down to coincide with the x -axis so that he could use his technique of factoring, which required the equation to be set equal to zero. In this instance, he was able to justify his process and not just execute it procedurally.

The influence of the Completing the Square investigation can also be seen in Jon's reference to the movement of the parabola's vertex during his verbal description of his graphical understanding. In this investigation, students were specifically asked to follow the movement of the vertex.

Jon clearly demonstrated an understanding of the concept of root and its connection to factoring as an equation-solving technique. He explained why he could not simply set each factor equal to zero in the equation $(x - 2)(x + 3) = 6$:

Jon: If I plug 2 into the first one, 2 minus 2 . . . that would give me 0, and 0 times anything would equal 0, not 6, so that's wrong. Then if I plug -3 in, -3 minus 2 would be -1, and then the second one [*referring to*

the second factor] would still be 0, so -1 doesn't equal 6 either [meaning entering -1 into the equation], so that would be wrong too.

Jon could articulate an understanding of the root-solving process using the zero property of multiplication along with his verbal reference to the vertex movement. This fact indicates an influence of both these investigations on his mathematical understanding and its relationship to the symbolic world that describes these ideas.

DYNAMIC LINKING IS THE KEY

From these contrasting examples, we see that, as we teach algebra, conceptual understanding can go hand-in-hand with procedural ability. Our research challenges the conventional wisdom that students must first become procedurally fluent before they can understand the concepts that we teach. Heid (1988) challenged this position more than two decades ago, arguing that we should rethink the sequencing of skills

and concepts in calculus by using computer algebra systems to develop concepts before teaching procedures.

Here we see this same principle applied to high school algebra concepts. The difference between our study and Heid's is that we suggest that it is not just the computer algebra system that influences how students see connections but rather the dynamic linking of various representations. Students in the first semester of this study used the TI-84 and were introduced to concepts before skills; however, they did not have the ability to manipulate various representations and see real-time effects among representations.

A second aspect of concept development that we noticed involves student control of the environment. As Lapp and Cyrus (2000) suggest from research on the use of data collection devices, the student's ability to manipulate the environment plays a significant role in making connections. During the first semester of this study, the teacher merely demonstrated some of the dynamically linked

representations using a computer during class; the students did not have use of this technology individually. Results of our interviews showed that none of these students could articulate connections among various representations. However, during the second semester, each student had a TI-Nspire CAS device and used it during student-centered investigations. This ability to communicate and analyze through dynamically connected representations between the symbolic world and the world of mathematical ideas allowed students to make connections between concepts and procedures.

As technology that links representations has become readily available, there is no reason we should not take advantage of it to better develop students' mathematical understanding. However, technology alone cannot make these connections for students. Kaput, Blanton, and Moreno (2008) as well as Sfard (2008) suggest that students' negotiation of discourse between the symbolic world and the world of mathematical ideas plays a key role in the development of symbolic

INSPIRING TEACHERS. ENGAGING STUDENTS. BUILDING THE FUTURE.

NCTM's Member Referral Program

Help Us Grow Stronger

Participating in **NCTM's Member Referral Program** is fun, easy, and rewarding. All you have to do is refer colleagues, prospective teachers, friends, and others for membership. Then as our numbers go up, watch your rewards add up.

Learn more about the program, the gifts, and easy ways to encourage your colleagues to join NCTM at www.nctm.org/referral. Help others learn of the many benefits of an NCTM membership—Get started today!



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

(800) 235-7566 | WWW.NCTM.ORG



meaning. Therefore, as teachers, we need to use appropriate technology to engage students in investigations that allow them to make mathematically meaningful observations and justify them.

ACKNOWLEDGMENTS

This research was supported in part with funding from the National Science Foundation (NSF) grant no. DMS 06-36528. The opinions expressed are those of the authors and do not necessarily represent the views of the National Science Foundation.

REFERENCES

- Beichner, Robert J. 1990. "The Effect of Simultaneous Motion Presentation and Graph Generation in a Kinematics Lab." *Journal of Research in Science Teaching* 27 (8): 803–15.
- Brasell, Heather. 1987. "The Effect of Real-Time Laboratory Graphing on Learning Graphic Representations of Distance and Velocity." *Journal of Research in Science Teaching* 24 (4): 385–95.
- Heid, M. Kathleen. 1988. "Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool." *Journal for Research in Mathematics Education* 19 (1): 3–25.
- Kaput, James J., Maria L. Blanton, and Luis Moreno. 2008. "Algebra from a Symbolization Point of View." In *Algebra in the Early Grades*, pp. 19–55. New York: Lawrence Erlbaum Associates, National Council of Teachers of Mathematics.
- Lapp, Douglas A. 1997. "A Theoretical Model for Student Perception of Technological Authority." In *Third International Conference on Technology in Mathematics Teaching*, vol. 29. Koblenz, Germany.
- Lapp, Douglas A., and Vivian Flora Cyrus. 2000. "Using Data-Collection Devices to Enhance Students' Understanding." *Mathematics Teacher* 93 (6): 504–10.
- Lapp, Douglas A., Melvin A. Nyman, and John S. Berry. 2010. "Student Connections of Linear Algebra Concepts: An Analysis of Concept Maps." *International Journal of Mathematical Education in Science and Technology* 41 (1): 1–18.
- Sfard, Anna. 1991. "On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin." *Educational Studies in Mathematics* 22 (1): 1–36.
- . 2008. *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. Cambridge: Cambridge University Press.
- Tall, David, and Shlomo Vinner. 1981. "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity." *Educational Studies in Mathematics* 12 (2): 151–69.

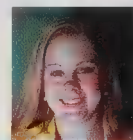


DOUGLAS A. LAPP,

lapp1da@cmich.edu, is professor of mathematics and mathematics education at Central Michigan University (CMU) in Mt. Pleasant. He is interested in the use of technology for the development of algebraic thinking and symbolic meaning.



MARIE ERMETE, ermet1mn@gmail.com, a senior at CMU, plans to teach high school mathematics. She is interested in the role of dynamic representations in students' development of mathematical concepts and



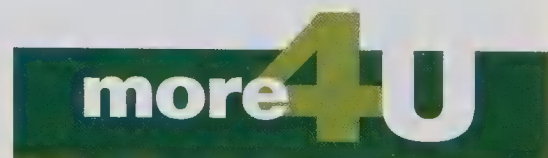
in mathematical fiber works. **NATASHA BRACKETT**, brack2nd@cmich.edu, also a senior at CMU, plans to teach middle school mathematics. She is interested in symbol development in middle school students. **KARLI POWELL**, kpowell@lindenschools.org, teaches at Linden High School in Linden, Michigan. She continues to integrate technology into her classroom to help students form connections across different representations and create a student-centered learning environment. Ermete, Brackett, and Powell were participants in the NSF-funded Long-term Undergraduate Research Experience (LURE) program at CMU.

I ♥ logarithmic spirals.

MATHEMATICS
IS ALL AROUND US.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS



To view and interact with some of the student investigations, download one of the free apps for your smartphone and then scan this tag to access www.nctm.org/mt044.



An Archimedean Balance: Polygons on the Head of a Pin

Archimedes's famous exposition "Mechanical Method" (Boyer 1991, pp. 136–39), in which he provides the "mechanical" derivation of the area formula for a parabolic sector, is one of the great pieces of mathematical literature. In it, Archimedes uses the law of the lever, along with a triumphantly sophisticated bit of calculus two thousand years before it was invented ("uses" is the wrong word, unless you think of Roger Federer as "using" a tennis racket or Picasso as "using" a paintbrush).

Since reading this exposition years ago, I have been intrigued by the potential interplay between pure mathematical theorems and physical properties. The traditional relationship between the two is to have mathematics describe and justify physical truths, which usually happens in physics class. But at times the process is reversed—that is, physical truths can be used as postulates to derive mathematical ones. I will illustrate this phenomenon with a small example of similar reasoning, showing its accessibility to high school students and its place in a mathematics classroom.

Years before I read Archimedes's derivation, I became fascinated by the purely mathematical properties of special points in triangles, but I was only mildly interested to learn that the centroid is a triangle's *balancing point*, the point at which a uniformly thick and dense triangle would balance on a pin (a key fact in Archimedes's parabolic sector derivation). But now the idea intrigued me. I wondered, How can we logically justify its truth? And how can we deduce the location of the balancing point for a more complicated polygon?

SOME POSTULATES

I eventually found answers using four simple, "mechanical" postulates, two of which refer to a *balancing line*, a line on which a polygon will balance as it would on a tightrope or the long edge of ruler. The postulates follow:

Postulate 1: The midpoint of a segment is its balancing point.

Postulate 2: Every polygon has a unique balancing point.

Postulate 3: Given a balancing line for a polygon, the balancing point lies on it.

Postulate 4: If a polygon is cut by a straight line into a pair of nonoverlapping polygons, then the (two) balancing points for those polygons define a balancing line for the original polygon.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more background information on the department and guidelines for submitting a manuscript, visit <http://www.nctm.org/publications/content.aspx?id=10440#delving>.

Edited by **Maurice Burke**, Maurice.Burke@utsa.edu
University of Texas at San Antonio

J. Kevin Colligan, jcolligan@verizon.net
SRA International, Columbia, MD (retired)

Maria Fung, mfung@worchester.edu
Worcester State College, Worcester, MA

Jeffrey J. Wanko, wankoji@muohio.edu
Miami University, Oxford, OH

The first three of these postulates are pretty straightforward, but you might have trouble believing the fourth.

To understand postulate 4, imagine cutting a polygon into two pieces, P_1 and P_2 , and calling the new edges that result from the cut the *shared sides*. Now suppose that we find the balancing point for each piece. Imagine balancing the pieces on the tips of two pins so that the shared sides are barely touching. At this moment, you are holding a pin in each hand; P_1 and P_2 are balanced on top of the pins; and the polygons themselves fit together like jigsaw pieces into the shape of the polygon before the cut.

Now imagine magically sealing the cut. This act has no effect on how anything balances; nothing moves. Now, what would happen if we placed a straight line (such as a tightrope or the edge of a ruler) through those two balancing points, pulled away the pins, and left the polygon on top of the straight line? I'm hoping that you see that the answer is *nothing happens*. The polygon balances on the line because it balanced on two points that were on that line. That's postulate 4.

TRIANGLES

Armed with our postulates, we can answer my two earlier questions: Why is the centroid a triangle's balancing point? And how can we locate the balancing points for other polygons? We need a lemma, and its proof relies on a tiny bit of nonrigorous calculus (which is certainly in the spirit of Archimedes).

Lemma 1: The median of a triangle is a balancing line for the triangle.

Proof: In $\triangle ABC$, let the altitude to side BC be h (see **fig. 1**). By postulate 1, the midpoint M of BC is the segment's balancing point. Let n be some large whole number and consider segments stretching from side AB to side AC , each parallel to side BC and each h/n distant from the segments immediately "above" and "below" it. The

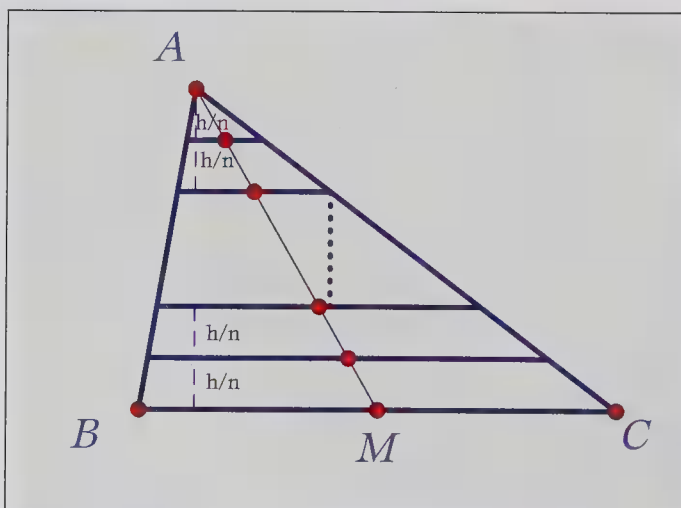


Fig. 1 To prove lemma 1, we slice a triangle with an increasingly large number of equally spaced cuts parallel to a side.

midpoints of these segments are their balancing points (postulate 1). As n approaches infinity, we see that the triangular region is this limiting collection of parallel segments, all of them balancing at the midpoints that form median AM . Because each segment balances on a point on AM , the triangle itself balances on AM . The lemma makes the answer to my first question obvious.

Theorem 1: The centroid of a triangle (i.e., the intersection of the medians) is the balancing point for the triangle.

Proof: Consider two medians of $\triangle ABC$. By lemma 1, they are balancing lines for ABC . By postulate 3, each of these lines contains the triangle's unique balancing point (unique by postulate 2), so the three lines' intersection is that point.

A nice by-product of this proof is the concurrence of the three medians. All we have to do is notice that the third median of ABC must, by lemma 1, also contain the unique balancing point. Therefore, the third median must pass through the point of intersection of the other two medians, proving concurrency. Physical properties prove this mathematical theorem.

QUADRILATERALS

We now move on to the harder question of how to find the balancing point for a polygon with more than three sides. It seems prudent to start small and consider a quadrilateral. (All polygons will be convex here, although it would be interesting to extend our discussion to nonconvex polygons.) At this point, when I have asked students to formulate a solution, even the brightest are tempted to make loose analogies that amount to guesswork. Sometimes guessing is a good idea, of course. But this is the moment to use rigorous mathematical reasoning to arrive at a solution. Here is a way to think through the problem.

The key to finding a polygon's balancing point is to recall a crucial observation from our last proof: The balancing point is at the intersection of two (or more) balancing lines. This makes our method clear: We will find a balancing line for the quadrilateral and repeat the process to find a second balancing line; our balancing point will be at their intersection. And how do we get a balancing line? We use postulate 4. (By the way, getting a group of students to think hard, discuss ideas deeply, and then articulate a plan like this is a beautiful thing to behold.)

Construction 1: To locate the balancing point of a quadrilateral.

Consider quadrilateral $ABCD$ in **figure 2**. We construct diagonal AC , dividing $ABCD$ into two

triangles. We locate the balancing points for $\triangle ABC$ and $\triangle ACD$ by intersecting medians (not pictured), giving us centroids G_{ABC} and G_{ACD} . By postulate 4, the line through the centroids is a balancing line for $ABCD$. Repeating this process, we now construct diagonal BD , dividing $ABCD$ into $\triangle ABD$ and $\triangle BCD$ with centroids G_{ABD} and G_{BCD} , the two of which define a second balancing line for $ABCD$. The intersection G of the balancing lines is the balancing point for $ABCD$. (G stands for the center of gravity.)

Again, a note about students' experience: If they are armed with good rulers (or straightedges and compasses, along with the knowledge of how to use them) and thin-tipped colored pencils, articulating construction 1's solution can provoke a huge amount of enthusiasm for testing out the "theory." I put the word in quotes because there is nothing theoretical about it. It *has* to work—that is, if the postulates are physically "true" and, more subtly, if the world of physical interactions is truly built on logic.

POLYGONS WITH MORE THAN FOUR SIDES

The next question is, What about polygons with more sides? Again, at this point, it can be instructive to watch students come up with spurious arguments, based mostly on loose analogies rather than logical thinking. Generally, one student offers something like, "Well, we cut the quadrilateral into two triangles and found their balancing points, so, *logically*, we should cut the pentagon into three triangles, find those balancing points . . . um . . . and they'll form a triangle . . . and, um . . . the centroid for that triangle is the balancing point for the original pentagon . . . ?" Logically? The answer is no. If students demand of themselves clear understanding of how the last construction worked, they can usually come up with the real key: The balancing point is at the intersection of a pair of balancing lines. And postulate 4 tells us how to construct these lines.

Construction 2: To locate the balancing point of a pentagon.

In **figure 3**, notice that segment AC cuts pentagon $ABCDE$ into two pieces, $\triangle ABC$, and quadrilateral $ACDE$. Using our previous discoveries, we find balancing points for each piece and label them G_{ABC} and G_{ACDE} . They define a balancing line. In addition, we construct AD , dividing the pentagon into $\triangle ADE$ and quadrilateral $ABCD$, themselves having balancing points labeled as G_{ADE} and G_{ABCD} . The line through those points gives us a second balancing line for $ABCDE$, and the intersection of the two balancing lines gives us G , the balancing point for the pentagon $ABCDE$.

This last construction does illustrate a recursive method for constructing the balancing point for any

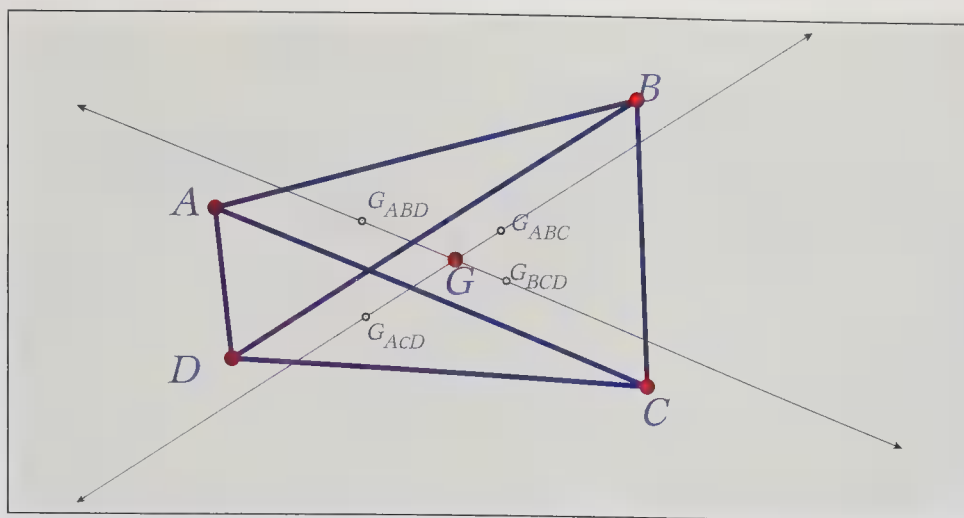


Fig. 2 Finding the balancing point of $ABCD$ reduces to determining two of its balancing lines.

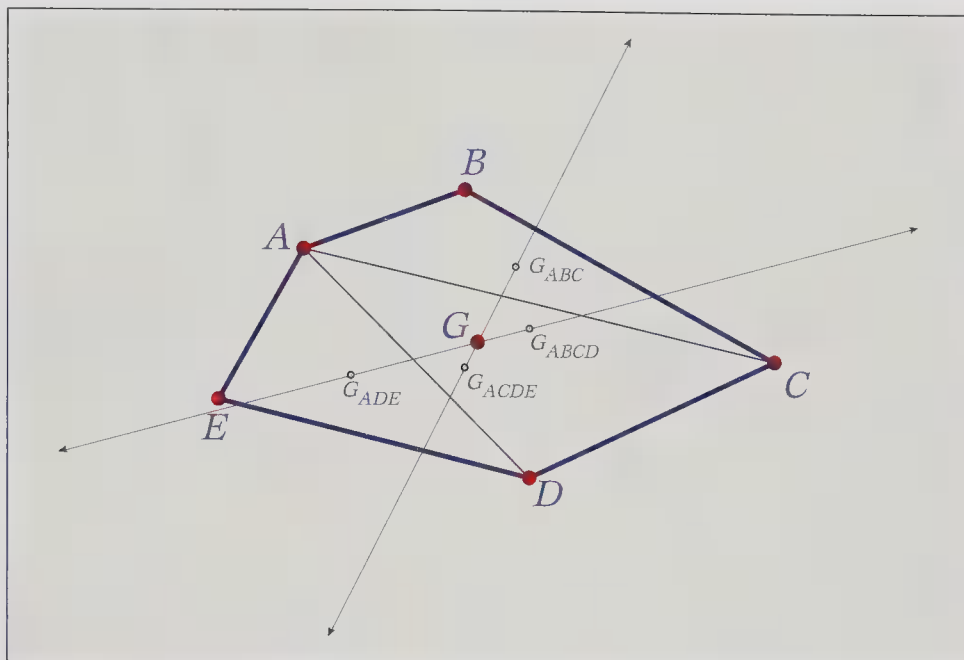


Fig. 3 To find the balancing point for a pentagon, we again find the intersection of two distinct balancing lines.

convex n -gon. In general, suppose that we know how to find the balancing point for some n -gon. With that knowledge, we can find the balancing point for an $(n + 1)$ -gon that we can name $ABCD$ Constructing segment AC cuts the polygon into an n -gon and a triangle. We find the balancing point for each, which themselves define a balancing line. Repeat with another cut (BD is eligible), and we get a second balancing line. The intersection of the balancing lines is the balancing point for the $(n + 1)$ -gon. This process for an arbitrary polygon can become increasingly tedious, but there is no doubt that it can be done because this line of reasoning serves the basis for a proof by mathematical induction.

Our last construction shows a pure mathematical theorem that can be proved by mechanical means. This is one of my favorite outcomes because this interplay is one of Archimedes's greatest and most singular insights. Consider the following mathematical (not physical) problem.

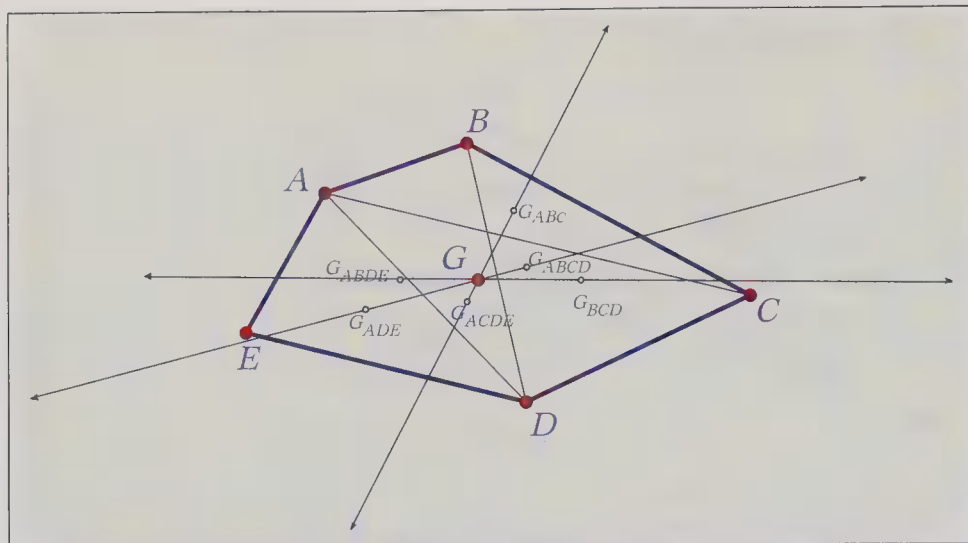


Fig. 4 Three concurrent lines are pictured, but many more are easily established by the same method.

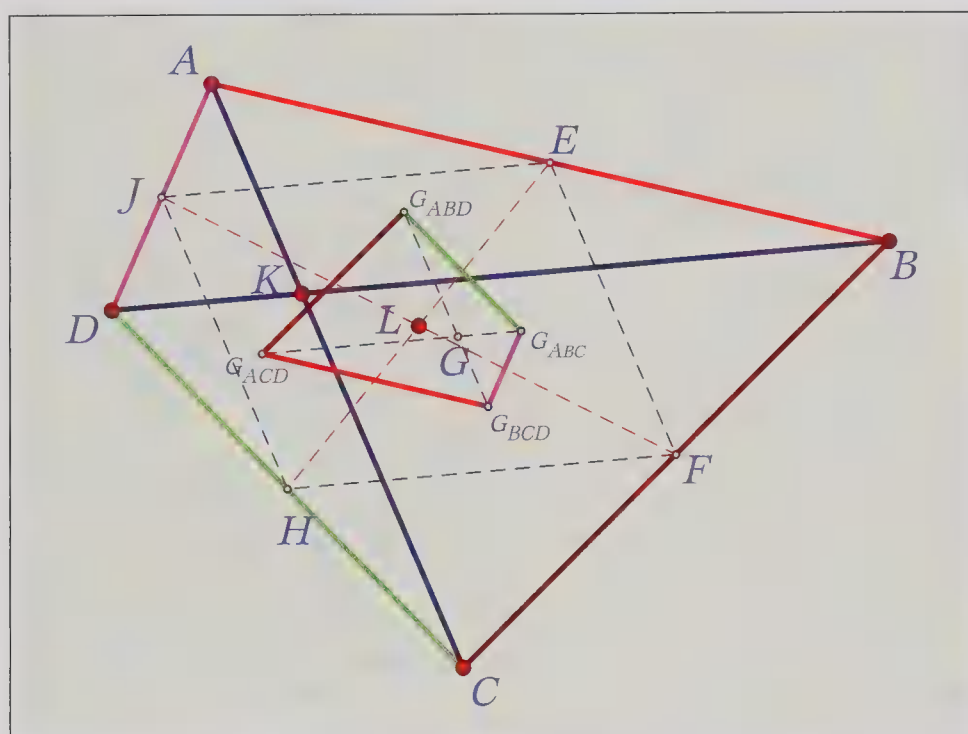


Fig. 5 Joe's theorem elegantly locates the balancing point of a quadrilateral.

Problem 1: Referring to the balance point notation that we have used, prove in pentagon $ABCDE$ that the lines $G_{ABC}G_{ACDE}$, $G_{ADE}G_{ABCD}$, and $G_{BCD}G_{ABDE}$ are concurrent.

Solution: By postulate 2, there is exactly one balancing point, G , for $ABCDE$. By postulate 4, each line in **figure 4** is a balancing line, so, by postulate 3, they all pass through G .

In fact, there are as many lines passing through G as there are ways to divide the pentagon into nonoverlapping triangle-quadrilateral pairs—in this case, five lines. In general, it is not hard to prove that there are n such concurrent lines through the balancing point of any n -gon and even more than that if we relax the requirement of cutting the n -gon into a triangle and $(n-1)$ -gon but instead allow cuts that result in, say, quadrilaterals and $(n-2)$ -gons. The mechanics of balancing has given us a purely mathematical result.

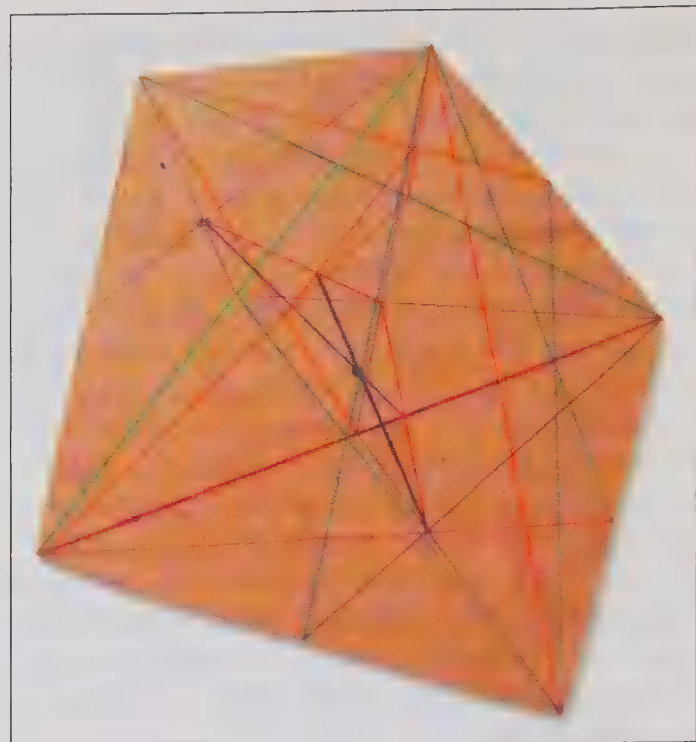


Fig. 6 The algorithm for constructing the balancing point for a pentagon leads to a mysterious network of line segments.

In conclusion, I relate a clever observation a student made about a quadrilateral's balancing point. A couple of months after studying balancing points, I gave the class a problem.

Problem 2: Given a quadrilateral $ABCD$, prove that the centroids of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$ form a quadrilateral that is a dilation of $ABCD$. Establish where the center of the dilation lies and the value of the dilation factor. (Rather than solve this problem here, I invite you to try it. It's a fun one. But here's a spoiler alert: I'm about to give away the answer, although not the solution).

Answer: If we label points as in **figure 2** and let E , F , H and J be midpoints of the respective sides of the original quadrilateral, then the center of dilation for $ABCD$ and the "centroid quadrilateral" $G_{BCD}G_{ACD}G_{ABD}G_{ABC}$ is the intersection point L of EH and FJ . The dilation factor of $ABCD$ to the centroid quadrilateral is -3 .

That day, after the class discussed the solution, I was ready to move on to something else. But one student, Joe, raised his hand: "That means that the balancing point of any quadrilateral $ABCD$ is a $4:3$ dilation of the point L if we use the intersection of the diagonals AC and BD as the center of dilation. That's a really easy construction!" No one else had thought of balancing points in connection with the problem. We were truly in awe of Joe's discovery. Here are the details of his argument.

Joe's theorem: Let the sides of quadrilateral $ABCD$ have midpoints at E , F , H , and J , as labeled in

figure 5, with L the intersection of $\overline{EH}$ and $\overline{FJ}$ and with K the intersection of diagonals AC and BD . The balancing point for $ABCD$ lies at the point G that is the result of a 4:3 dilation about center K .

Proof: We know from construction 1 that the balancing point of $ABCD$ lies at the intersection point, G , of the diagonals of quadrilateral $G_{BCD}G_{ACD}G_{ABD}G_{ABC}$ (see **fig. 5**). By the answer to problem 1, $ABCD$ and $G_{BCD}G_{ACD}G_{ABD}G_{ABC}$ are dilations of each other about point L . Now consider points K and G . Because they are intersections of the diagonals of these quadrilaterals that correspond in a dilation, we conclude that K and G themselves are dilations of each other about point L . (This was the essential piece of Joe's brilliant insight.) So by problem 2's answer, G is the result of a dilation by a factor of -3 about point L . This is equivalent to saying that G is the result of a 4:3 dilation about center K .

Joe's theorem is wonderful for lots of reasons, but one nice outcome is how it makes construction 2 simpler to accomplish. It is also aesthetically a pleasing location for the balancing point.

One final thought: Until you do it, you really don't know how much fun it is to draw a bunch of crazy-looking lines on a piece of poster board



Fig. 7 The pentagon balances right on the dot.

(see **fig. 6**) and then watch the thing balance on the flat tip of a chopstick (better than a pin, trust me) (see **fig. 7**). I highly recommend it.

REFERENCE

Boyer, Carl B. 1991. *A History of Mathematics*. 2nd ed. New York: John Wiley and Sons.



CHARLES WORRALL, charles_worrall@horacemann.org, is the chair of the Upper Division Mathematics Department at Horace Mann School, Bronx, New York.

He loves teaching an honors geometry course, reading particularly elegant mathematics, and having an occasional "math night" with his two daughters.

Statement of Ownership, Management, and Circulation

Statement of ownership, management, and circulation (Required by 39 U.S.C. 3685). 1. Publication title: *Mathematics Teacher*. 2. Publication number: 334-020. 3. Filing date: August 19, 2013. 4. Issue frequency: Monthly: August–December/January; February–May. 5. Number of issues published annually: 9. 6. Annual subscription price: \$34. 7. Complete mailing address of known office of publication: National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Fairfax County, VA 20191-1502. Contact person: Pamela Ann Grainger, (703) 620-9840, ext. 2167. 8. Complete mailing address of headquarters or general business office of publisher: same as #7. 9. Full names and complete mailing addresses of publisher, editor, and managing editor. Publisher: National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Fairfax County, VA 20191-1502. Editor: Pamela Ann Grainger, 1906 Association Drive, Reston, VA 20191-1502. Managing editor: none. 10. Owner: National Council of Teachers of Mathematics (nonprofit organization), 501(c)3, 1906 Association Drive, Reston, Fairfax County, VA 20191-1502. 11. Known bondholders, mortgagees, and other security holders owning or holding 1 percent or more of total amount of bonds, mortgages, or other securities: none. 12. Tax status. The purpose, function, and nonprofit status of this organization and the exempt status for federal income tax purposes has not changed during preceding 12 months. 13. Publication title: *Mathematics Teacher*. 14. Issue date for circulation data below: July 24, 2013. 15. Extent and nature of circulation. Average no. copies each issue during preceding 12 months. A. Total number of copies: 20,722. B. Paid circulation. (1) Mailed outside-county paid subscriptions stated on PS form 3541: 18,649; (2) mailed in-county paid subscriptions stated on PS form 3541: none; (3) paid distribution outside the mails including sales through dealers and carriers, street vendors, counter sales, and other paid distribution outside USPS: 800; (4) paid distribution by other classes of mail through the USPS: none. C. Total paid distribution: 19,449. D. Free or nominal rate distribution: (1) free or nominal rate outside-county copies included on PS form 3541: none; (2) free or nominal rate in-county copies included on PS form 3541: none; (3) free or nominal rate copies mailed at other classes through the USPS: 534; (4) free or nominal rate distribution outside the mail: 222. E. Total free or nominal rate distribution: 756. F. Total distribution: 20,205. G. Copies not distributed: 517. H. Total: 20,722. I. Percent paid: 96%. 15. Extent and nature of circulation. No. copies of single issue published nearest to filing date. A. Total number of copies: 19,630. B. Paid circulation. (1) Mailed outside-county paid subscriptions stated on PS form 3541: 17,931; (2) mailed in-county paid subscriptions stated on PS form 3541: none; (3) paid distribution outside the mails including sales through dealers and carriers, street vendors, counter sales, and other paid distribution outside USPS: 737; (4) paid distribution by other classes of mail through the USPS: none. C. Total paid distribution: 18,668. D. Free or nominal rate distribution: (1) free or nominal rate outside-county copies included on PS form 3541: none; (2) free or nominal rate in-county copies included on PS form 3541: none; (3) free or nominal rate copies mailed at other classes through the USPS: 440; (4) free or nominal rate distribution outside the mail: none. E. Total free or nominal rate distribution: 440. F. Total distribution: 19,108. G. Copies not distributed: 522. H. Total: 19,630. I. Percent paid: 98%. 16. Publication of statement of ownership will be printed in the November 2013 issue of this publication. 17. Signature and title of editor, publisher, business manager, or owner: Pamela Ann Grainger, senior copy editor, August 19, 2013. I certify that all information furnished on this form is true and complete. I understand that anyone who furnishes false or misleading information on this form or who omits material or information requested on the form may be subject to criminal sanctions (including fines and imprisonment) and/or civil sanctions (including civil penalties).

PUBLICATIONS

From NCTM

Individual NCTM members receive a 20 percent discount on NCTM publications. To order, visit the NCTM online catalog at www.nctm.org/catalog or call toll free at (800) 235-7566. Free print catalogs of NCTM publications also are available by writing to NCTM.

Professional Collaborations in Mathematics Teaching and Learning: Seeking Success for All: 74th Yearbook (2012),

Jennifer M. Bay-Williams and William R. Speer, eds., 2012. 309 pp., \$58.95 cloth. ISBN 978-0-87353-697-4. Stock no. 14316. National Council of Teachers of Mathematics; www.nctm.org.



Professional Collaborations in Mathematics Teaching and Learning was written to help teachers and teacher leaders effectively implement learning communities within their schools. The book aims to encourage discussions of formative and summative assessments and their roles in the classroom. Many issues must be resolved when developing a learning community, and this book helps provide solutions to many of these.

The book examines many school and university collaborations; as a result, there is something here for every reader, no matter his or her role. I especially liked the chapters that discussed the partnerships between universities and high schools and middle schools, and the section on Instructional Coaching

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

(part IV) was helpful. It is easy to get stuck doing the same things every day, and *Professional Collaborations* helped put into perspective the change that we can expect to see if teachers are working toward a common goal.

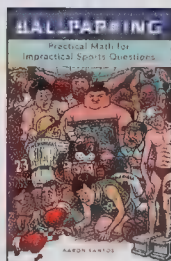
Although the book was helpful, it did take awhile for me to get through it. The amount of information was hard to digest and hard to remember by the time I had finished it. I will have to reread parts several times to get the most out of the material.

I highly recommend *Professional Collaborations* to teachers and teacher leaders. A great amount of this information will be useful as I help structure the learning communities that I will be a part of in the future.

—Melissa Keller
Ludlow Independent Schools
Cincinnati, OH

FROM OTHER PUBLISHERS

Ballparking: Practical Math for Impractical Sports Questions, Aaron Santos, 2012. 220 pp., \$15.00 paper. ISBN 978-0-7624-4345-1. Running Press; www.runningpress.com.



A back-of-the-envelope calculation is a wonderful thing. A person can take a few reasonable assumptions and a few well-chosen facts, apply dimensional arguments and sensible mathematics, and come up with surprising results. These results may not be precise, but they are the right order of magnitude—and often that is all we need. These problems, also called Fermi problems, make us mathematically powerful: We can play outrageous “What if?” games and can determine, quantitatively, what is possible and what is ridiculous.

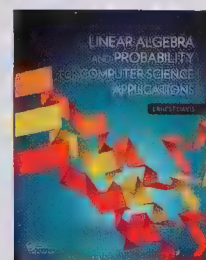
Ballparking is essentially a collection of these problems (and their solutions) inspired by sports and accessed largely through physics. The book is not intended as a guide for students or teachers but could be used as a resource. Santos explains his strategies, reminds us of his assumptions and approximations, and even includes sections on dimensional analysis and how to use scaling arguments—for example, how the range of a projectile scales inversely with gravitational acceleration.

Ballparking is also refreshingly funny and unselfconscious. Santos is a guy who loves sports and physics. But if you don't know sports-for-guys (there's not a woman athlete in the book), the references will pass you by. And—fair warning—some topics will offend some readers.

Unfortunately, the book was poorly proofread. There are missing words, bad typesetting, misplaced data, and broken equations. If you know what's going on, you can make the corrections yourself. But you shouldn't have to.

—Tim Erickson
Epistemological Engineering
Oakland, CA

Linear Algebra and Probability for Computer Science Applications, Ernest Davis, 2012. 431 pp., \$59.95 cloth. ISBN 978-1-4665-0155-3. CRC Press, Taylor and Francis Group; www.taylorandfrancis.com.



This book was written for a master's program course entitled Mathematical Techniques for Computer Science Applications, used to supplement the basic discrete mathematics course that computer scientists take. It is one of the few books that include both linear algebra and probability.

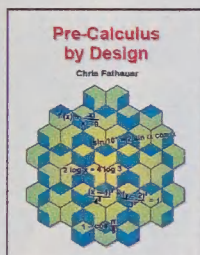
The course covers new mathematical areas of concern for the computer scientist, including artificial intelligence, graphics, optimization, and information retrieval. Linear algebra topics cover standard areas and are supplemented with MATLAB exercises on topics such as vectors. Probability topics include basic combinatorics, random variables, Markov models, confidence intervals, and Monte Carlo methods. Optional sections are helpfully marked as such, and the author assumes as little mathematics background as possible for the problems, sometimes only high school level.

I would recommend this book for computer science students needing content in both linear algebra and probability.

—Anne Quinn

Edinboro University of Pennsylvania

Pre-Calculus by Design, Chris Fathauer, 2012. Grades 10-12, \$19.95 paper. ISBN 978-0-9846042-7-2. Jacobs Publishing, Tessellations; www.mathartfun.com.



This book contains forty-two exercise sets designed to help students review pre-calculus concepts from linear equations and inequalities through

limits and a few basic calculus topics. In a novel twist, each set is accompanied by a visual grid keyed to correct responses to exercises. As students complete problems within each set, they shade designated portions of the grid to reveal a “pleasing symmetrical design.”

The variety of topics contained in *Pre-Calculus by Design* (one of sixteen titles in the By Design series) provides a comprehensive review of most topics in a typical precalculus or analysis course. The breadth of topics is impressive, allowing classroom teachers to select exercise sets appropriate to content coverage and level of challenge. Classroom testing of several exercise sets with small groups of students showed that the combination of mathematical content with tessellation design motivated students to complete the problems.

However, students noted several deficiencies in the book’s approach. The

problems within each set became repetitive, and reasonably astute students with some knowledge of symmetry were able to complete the designs without completing all the mathematical exercises, supporting the author’s suggestion of asking students to show work when material from this book is used in review situations.

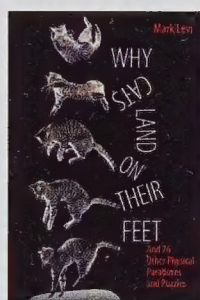
I recommend *Pre-Calculus by Design* as a supplemental resource for review and practice of selected topics in precalculus. In particular, it allows teachers to appeal to their students’ algorithmic as well as visual learning strengths.

—Jodie A. Miller

Mary Baldwin College

Staunton, VA

Why Cats Land on Their Feet: And 76 Other Physical Paradoxes and Puzzles, Mark Levi, 2012. 216 pp., \$19.95 paper. ISBN 978-0-691-14854-0. Princeton University Press; www.press.princeton.edu.



Cats land on their feet by squirming—as everyone knows—but what are the laws of physics that underlie this phenomenon?

It all has to do with conservation of angular momentum, as Mark Levi explains in this captivating book. The author explores many other equally puzzling curiosities with just a few laws of physics, including Newton’s laws, conservation of energy, and the conservation of linear momentum and angular momentum. (The laws of physics are elaborated in an appendix.)

The paradoxes and puzzles included here span a wide range: moving and falling objects (cars, bicycles, boats, balloons, satellites, corks, icebergs, and more) in space, on earth, and in water; fluid flow (in pipes, drains, sprinklers, rivers, and storms); heating and cooling; gyroscopes; and forces. The best part of all of this is the mathematics (of course); Levi gives straight, clear mathematics-based explanations for all kinds of strange behavior. Most of the puzzles could be adapted to the classroom, showing students what they can do with squares and square roots, algebra, geometry, vectors,

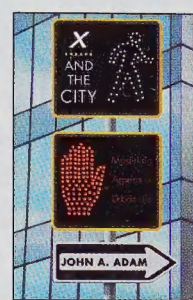
trigonometry, inequalities, and calculus. Many of the puzzles could be used as a basis for student projects.

This book will cultivate and challenge your physical intuition. Above all, it shows that physics and mathematics can be fun and useful at the same time.

—Catherine A. Gorini

Maharishi University of Management
Fairfield, IA

X and the City: Modeling Aspects of Urban Life, John A. Adam, 2012. 264 pp., \$24.95 cloth. ISBN 978-0-691-15464-0. Princeton University Press; www.press.princeton.edu.



X and the City applies mathematical modeling to situations that revolve around city living: Why does a tunnel cause traffic to bottleneck? What is the average distance traveled in a city center?

If all the taxis are busy and you must walk home in the rain, is it better to walk or run? What is the likelihood that your city will be struck by an asteroid?

Working with mathematics from precalculus through differential equations, the author moves from topic to topic, using mathematics to systematically understand aspects of life in a city. Because the unifying theme is city life, the mathematics runs the gamut from estimation, discrete and continuous modeling, probability, and geometry to mathematical physics. The author has an entertaining style, interweaving clever stories with the process of mathematical modeling.

This book is not designed as a textbook, although it could certainly be used as an interesting source of real-world problems and examples for advanced high school mathematics courses. Exercises are sprinkled throughout, and many fruitful mathematical discussions could arise from examining the assumptions and simplifications that the author makes for each model.

And don’t fret. If an asteroid is headed straight for Earth, the chance of your city being hit is less than 10^{-5} .

—Theresa Jorgensen

University of Texas at Arlington

MY FAVORITE lesson

Clarence W. Lienhard

Perfect Sample and Population Mean Formula

Students readily master some concepts of statistics. The sample mean formula

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n},$$

where n is the sample size and x_i are the observations, is accessible to students because they are familiar with averages and because, as a measure of location, it is easily observable. Likewise, a discrete random variable X with its probability mass function $p(x)$ and (finite) space A can be understood through simple, concrete examples such as rolling a polyhedron. However, students often have difficulty comprehending the population mean formula:

$$\mu = \sum_{x \in A} x \cdot p(x) \quad (1)$$

My favorite lesson uses the concept of a perfect sample to help students gain insight into this formula and understand the difference between population parameters and sample statistics. We begin with a discrete random variable X with $A = \{1, 2, 3, 4, 5\}$ and $p(x) = \{.10, .25, .45, .15, .05\}$, respectively.

My students easily answer my first question: What is the probability that we get a 1 when we sample this distribution? We record the correct response of 10%, and I then ask a different version of the same question: In a perfect sample, what percentage of 1s would we observe? Again, we record the correct response of 10%. From these opening questions, we begin the process of creating a perfect sample of size 200 from this distribution. (I remind students that such a sample, being perfect, is not what we usually see in practice.) We start by listing the 1s: Students easily calculate that we need 20 1s in our sample. Continuing, we get 50 2s, 90 3s, 30 4s, and 10 5s. Students record this information, creating a concrete listing to refer to as we work.

The class calculates the sample mean of our perfect sample. We find the numerical value, but, more important, we focus

on any relationship to $p(x)$ as we do the calculations. We separate the expression into five fractions with a common denominator of 200. Thus,

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} = \frac{1(20) + 2(50) + 3(90) + 4(30) + 5(10)}{200} \quad (2)$$

$$= \frac{1(20)}{200} + \frac{2(50)}{200} + \frac{3(90)}{200} + \frac{4(30)}{200} + \frac{5(10)}{200} \quad (3)$$

$$= 1(.10) + 2(.25) + 3(.45) + 4(.15) + 5(.05) = 2.80. \quad (4)$$

In equation (2), we connect our calculations to a weighted mean, a concept already familiar to students. When we get to (4), we discuss the place of $p(x)$ in our expression. At first, my students have difficulty articulating the connections they see, but together we get to a formula:

$$\sum x \cdot p(x)$$

Because we began with a perfect sample, the population's distribution is perfectly represented, and thus we must have calculated the population mean. Consequently, the formula we have created is that for the population mean, (1). We apply this new formula to several examples from other distributions.

I emphasize that when we take a perfect sample, $\bar{x} = \mu$; in practice, however, we almost never have a perfect sample, and so almost always $\bar{x} \neq \mu$. To reinforce this, we randomly simulate a sample of size 200 from $p(x)$ as given above and calculate $\bar{x}$. For one particular simulation, we saw a sample mean of $\bar{x} = 2.72$ and discussed how it compared with our population mean, $\mu = 2.80$.

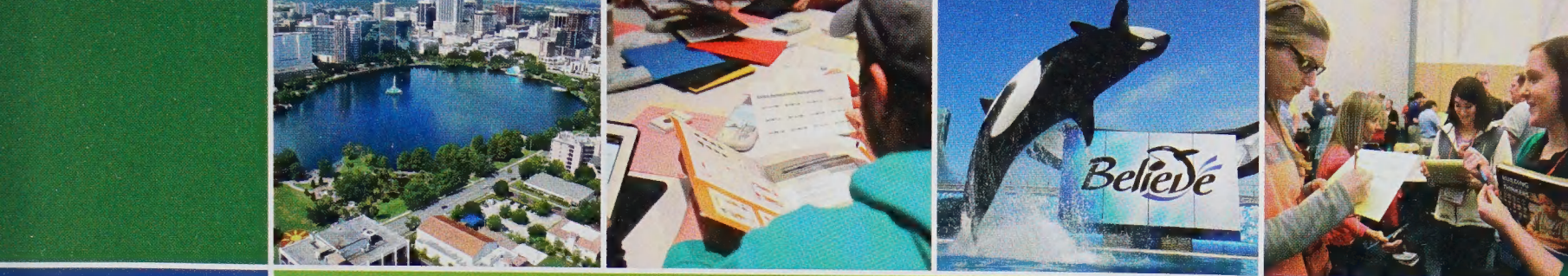
This perfect sample technique can be used in a variety of lessons, including understanding the random variable $\bar{X}$ and developing the population variance. This technique is an intuitive approach to probability and expectation that helps students connect new ideas to familiar material.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by **Jennifer Wexler**, wexlerj@newtrier.k12.il.us,
New Trier High School, Winnetka, IL



CLARENCE W. LIENHARD, clienhar@mansfield.edu, teaches in the Department of Mathematics and Computer Information Science at Mansfield University in Mansfield, Pennsylvania. He is interested in a wide range of topics, including probability, calculus, operations research, numerical analysis, and statistics.



February 14–15, 2014 | Orlando, FL

Cutting to the "Common Core" for Grades 9–12

THE NCTM INTERACTIVE INSTITUTE SERIES



Space
is limited—
**REGISTER
TODAY!**



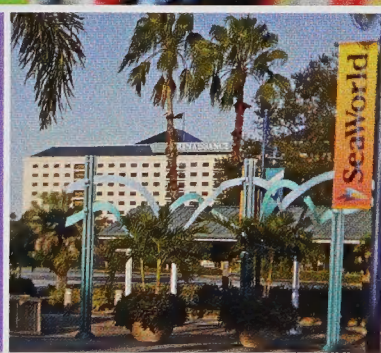
The Interactive Institute for Grades 9–12 Teachers of Mathematics

Increase your knowledge of mathematics content related to the Common Core domains for the high school grades, and learn strategies that will help you align your instruction with the Common Core State Standards for Mathematics.

NCTM's Interactive Institute offers the latest strategies to ensure that your students receive the best preparation for higher education and beyond.

Topics addressed in the NCTM Interactive Institute Series include the following:

- Algebra
- Functions
- Modeling
- Geometry



GROUP DISCOUNTS ARE AVAILABLE

This institute is part of our comprehensive series available for PK–12 teachers and school leaders.

Schools/Districts are encouraged to send groups to learn as a team.

Visit the website for more information.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

(800) 235-7566 | WWW.NCTM.ORG

SPACE IS LIMITED. Register early at
www.nctm.org/CCSSMHS to secure your spot.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

THE NATION'S PREMIER MATH EDUCATION EVENT

2014 ANNUAL MEETING & EXPOSITION

April 9–12 • New Orleans, LA

Big Ideas in the Big Easy!

Join us in New Orleans for the nation's largest math education event. More than 700 presentations will offer ideas, tools, and strategies you can immediately apply to help your students grow and succeed. Whether you're a classroom teacher, coach, administrator, teacher-in-training, or math specialist, NCTM's Annual Meeting has something for you.

- Learn practices central to teaching the **Common Core State Standards**.
- Gain practical solutions to transform your classroom into an environment rich in **problem solving**.
- Discover new and effective methods to incorporate **technology** in the classroom.
- Get answers to pivotal questions and concerns of **new and soon-to-be teachers**.

Helping students to develop essential math skills begins with you. This is the math education event you can't afford to miss!

Housing
is now open.

Registration
opens
November 1.

Visit www.nctm.org/neworleans for up-to-date information and follow us on

